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ABSTRACT

This is one in a series of manuals for teachers using SMSG high school supplementary materials. The pamphlet includes commentaries on the sections of the student's booklet, answers to the exercises, and sample test questions. Topics covered include sets, definition and graph of a function, constant, linear and absolute-value functions, composition, inversion, one-to-one functions, ordered pairs, circular motion, graphs of sine and cosine, angles, vectors, addition formulas, tables of circular functions, and waves. (MP)

SCHOOL
MATHEMATICS
STUDY GROUP

SP-3

SUPPLEMENTARY and ENRICHMENT SERIES

FUNCTIONS

CIRCULAR FUNCTIONS

Teachers' Commentary

Edited by Roy Dubisch

SMSG

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PREFACE

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which, though within the grasp of secondary school students, do not find a place in the curriculum simply because of a lack of time.

Many classes and individual students, however, may find time to pursue mathematical topics of special interest to them. This series of pamphlets, whose production is sponsored by the School Mathematics Study Group, is designed to make material for such study readily accessible in classroom quantity.

Some of the pamphlets deal with material found in the regular curriculum but in a more extensive or intensive manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum. It is hoped that these pamphlets will find use in classrooms in at least two ways. Some of the pamphlets produced could be used to extend the work done by a class with a regular textbook but others could be used profitably when teachers want to experiment with a treatment of a topic different from the treatment in the regular text of the class. In all cases, the pamphlets are designed to promote the enjoyment of studying mathematics.

Prepared under the supervision of the Panel on Supplementary Publications of the School Mathematics Study Group:

Professor R. D. Anderson, Louisiana State University

Mr. M. Philbrick Bridgess, Roxbury Latin School, Westwood, Massachusetts

Professor Jean M. Calloway, Kalamazoo College, Kalamazoo, Michigan

Mr. Ronald J. Clark, St. Paul's School, Concord, New Hampshire

Professor Roy Dubisch, University of Washington, Seattle, Washington

Mr. Thomas J. Hill, Oklahoma City Public Schools, Oklahoma City, Okla.

Mr. Karl S. Kalman, Lincoln High School, Philadelphia, Pennsylvania

Professor Augusta L. Schurrer, Iowa State Teachers College, Cedar Falls

Mr. Henry W. Syer, Kent School, Kent, Connecticut

TEACHER'S COMMENTARY

This pamphlet includes commentaries on the SMSG pamphlets <u>Functions</u> and <u>Circular Functions</u>. In addition to comments on the various sections, there are included solutions to all exercises and a set of illustrative test questions for each pamphlet.

TEACHER'S COMMENTARY

FOR

FUNCTIONS AND CIRCULAR FUNCTIONS

FUNCTIONS

CIRCULAR FUNCTIONS

FUNCTIONS

1. Sets.

It should be emphasized that we are not teaching "set theory" here. We are only concerned with the basic intuitive concepts of a set and the language used in talking about sets.

Answers to Exercises 1

In all of the following answers, letters, symbols, and names of people or objects, as well as their sequence, may be different without making the answers incorrect. It is not necessary always to name a set by means of a capital letter.

1. (a) $V = \{a, e, i, o, u\}$ or $V = \{*:\ *$ is a vowel$\}$

   (b) $P = \{2, 3, 5, 7, 11, 13, 17, 19\}$

   $= \{p:\ p$ is a prime number less than $20\}$

   For technical reasons, 1 is not considered a prime number. For example, its inclusion would raise a difficulty in the unique factorization theorem.

   (c) $R = \{$(insert names of people living in your house)$\}$

   $= \{a:\ a$ is a person who lives in my house$\}$

   (d) $T = \{3, 9, 15, 21\}$

   $= \{n:\ n$ is an odd multiple of 3 and $n \leq 21\}$

   (e) $N = \{17, 26, 35, 44, 53, 62, 71, 80\}$

   $= \{x:\ x$ is a two-digit integer, the sum of whose digits is $8\}$

   Note: 08 is not considered a two-digit number in our system.

2. (a) $S = \{s:\ s$ is a student in our school$\}$

   (b) $M = \{\#:\ \# > 7\}$

   (c) $P = \{p:\ p$ is a person in our community who found a ten-dollar bill yesterday$\}$

   Note that P may be the empty (or null) set with no elements.

   (d) $B = \{b:\ b$ is a book in our school library$\}$

   (e) $F = \{f:\ f$ is a rational number between 2 and $3\}$

   Here are three types of sets which are not tabulated.

   (a) and (d) represent extensive and lengthy lists which are available somewhere as completely tabulated sets but usually not duplicated.

   (b) and (e) represent examples of sets which contain an endless number of elements and thus defy listing.

(c) represents a condition frequently found in mathematics, where even though the description is clear and well defined it still requires a great deal of work or ingenuity to find the elements.

3. (a) $A = \{a:\ a$ is a positive even integer less than $12\}$ or
$\{a:\ a$ is an even natural number less than or equal to $10\}$
(b) $B = \{b:\ b$ is an integer whose square is less than $10\}$ or
$\{b:\ b$ is an integer and $-3 \leq b \leq 3\}$
(c) $C = \{c:\ c$ is the square of $1, 2, 3, 4,$ or $5\}$ or
$\{c:\ c$ is the square of an integer and $0 < c < 26\}$ or
$\{c^2:\ c$ is an integer and $1 \leq c \leq 5\}$
(d) $D = \{d:\ d = 2 + 3n,\ n$ is an integer, and $0 \leq n \leq 5\}$ or
$\{d:\ d$ is a number of the form $3n - 1$, and $n = 1, 2, 3, 4, 5,$ or $6\}$ or
$\{d:\ d$ is a term in an arithmetic sequence whose first term is 2, whose common difference is 3, and whose last term is $17\}$
(e) $E = \{e:\ e$ is a permutation of the digits $1, 2,$ and $3\}$ or
$\{abc:\ abc$ is a permutation of $123\}$ or
$\{e:\ e$ is a three-digit integer formed from the digits $1, 2,$ and 3 without repetition$\}$

---

2. Definition of Function.

A function can be defined in a variety of ways. The definition we have given was selected because it emphasizes the modern point of view of a function as a mapping, a point of view which is a particularly useful one in describing the composition of functions and inverse functions. Another common contemporary practice is to define a function as a set of ordered pairs in which no two distinct pairs have the same first component; this definition is a particularly convenient one in dealing with graphs. This point of view is described in Section 9.

You should be very careful at this stage to insist upon the proper use of functional notation and how to read it. If we write

$$f:\ x \rightarrow y$$

then we read,

"The function f that takes (or maps) x into y."

If we write $y = f(x)$, we read

"y is the value of f at x."

The student should not be permitted to say that $y = f(x)$ is a function. This is a common error of usage. Many mathematicians still use $y = f(x)$ elliptically, but being mathematicians, they understand what they are doing. High school students, however, are apt to be very confused by this, and we wish to do everything we can to be clear about the matter. Thus, $y = 3 - x$ is not a "linear function" although it may be used to define one over the real numbers, i.e.,

$$f: \ x \rightarrow 3 - x.$$

In explaining the function concept, you will probably wish to make use of a variety of techniques. The representation as a machine is one. Another approach might be to suggest a function from a domain consisting of the students in the class to a range consisting of the seats in the class and then ask for restrictions on the assignment so that it represents a function. For instance, two different seats could not be assigned to the same student; at least one seat would have to be assigned to each student, etc. Or again, inquire into the possibility of defining a function from the set of students to the set of their weights. Such examples are easy to devise and provide a means of focusing attention on the essential properties of a function. Also, it is useful for the student to be aware of the fact that the domain and range of a function need not be numerical. Many times it is useful to consider a function from the real numbers, say, to a set of points in a plane or vice versa. (See, for example, the SMSG pamphlet Circular Functions.)

It is also helpful to use examples from the sciences. You might ask the students what physicists mean when they say that the length of a metal bar is a function of the temperature of the bar, or that the pressure of a gas at a given temperature is a function of the volume it occupies. Make sure in each case that the students arrive at a function from the real numbers to the real numbers.

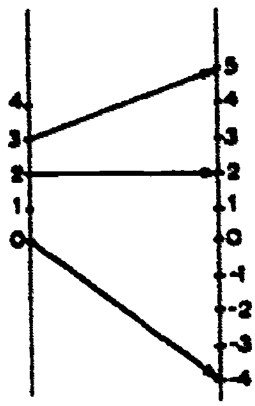
When introducing the idea of a function as a mapping, you should emphasize the point that there cannot be more than one arrow from each element of the domain, while there can be any number of arrows to each element of the range. If there is just one arrow to each element in the range, then the function is said to be one-to-one and, as will be seen later, has an inverse.

The concepts of domain and range should be emphasized. It should be made clear that in order to define a function, we must have a domain. (It will prove valuable to the students if, from time to time after you have completed the unit, you stop and ask for the domain and range of whatever function you may be considering at the time.)

Answers to Exercises 2

1. d, because it is multiple valued

2. (a)



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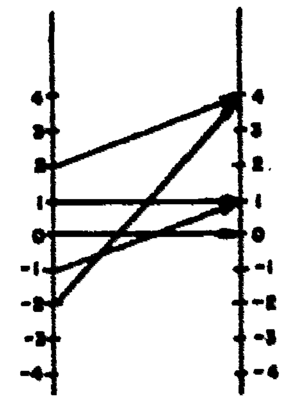
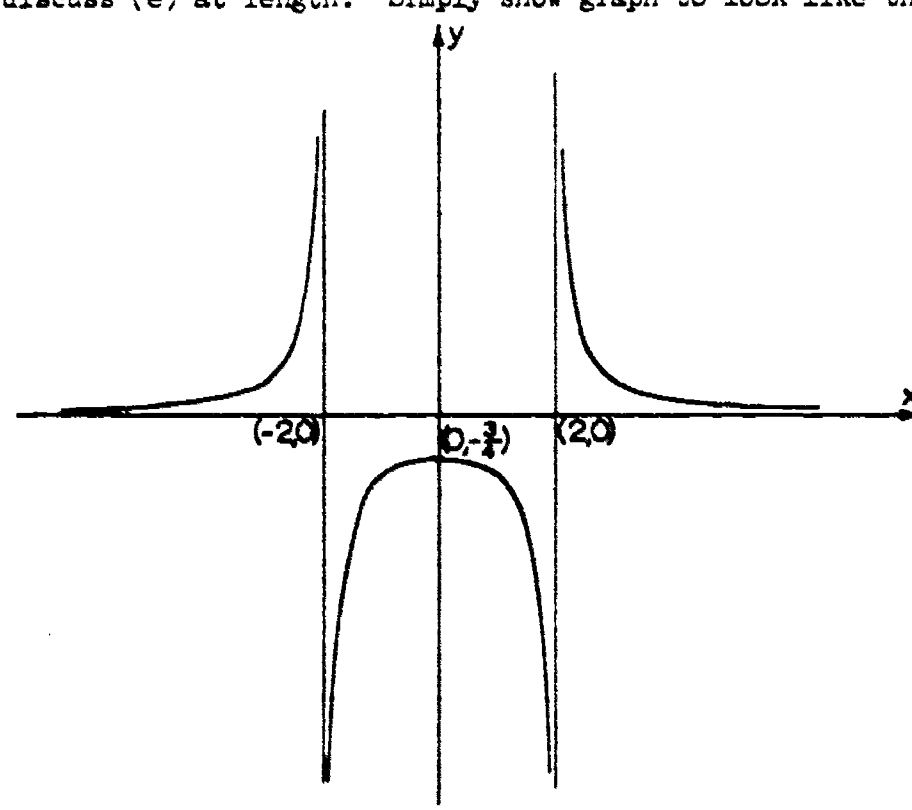


3.

	Domain	Range
(a)	R	R
(b)	R	non-negative R
(c)	non-negative R	non-negative R
(d)	R except $x = 1$	R except $f(x) = 1$
(e)	R except $x = 2$ or -2	R except $-\frac{3}{4} < f(x) < 0$

To find the range in (d), set $y = \frac{x}{x - 1}$ and solve for x: $x = \frac{y}{y - 1}$. This shows that $y \neq 1$.

Do not discuss (e) at length. Simply show graph to look like this:



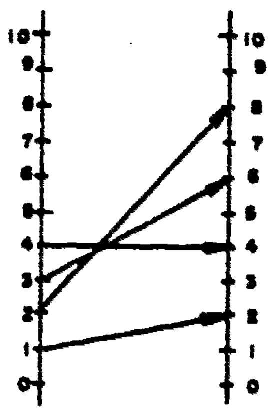
If you test to find values of x for $\frac{3}{x^2 - 4}$ equal to numbers between $-\frac{3}{4}$ and 0, you will obtain imaginary values.

4. (a) $f(0) = 1$ (c) $f(100) = 201$
   (b) $f(-1) = -1$ (d) $f(\frac{3}{2}) = 4$

5. (a) $f(0) = 3$ (c) $f(a) = a^2 - 2a + 3$
   (b) $f(-1) = 6$ (d) $f(x - 1) = x^2 - 4x + 6$

6. (a) $f(4) = 0$ (d) $f(a) = \sqrt{a^2 - 16}$
   (b) $f(-5) = 3$ (e) $f(a - 1) = \sqrt{a^2 - 2a - 15}$
   (c) $f(5) = 3$ (f) $f(\pi) = \sqrt{\pi^2 - 16}$

7. $D = \{1, 2, 3, 4\}$ $R = \{2, 4, 6, 8\}$



8. They are not the same function, since g does not include 0 in its domain.

9. (a) 4,-4 (b) 8 (c) 12,-12

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3. The Graph of a Function.

The graph is perhaps the clearest means of displaying most functions since the story is all there at once. The student can observe the behavior of f for the various portions of the domain, and, in most cases, irregularities are obvious immediately. The difficulty is, of course, that the graphs of some functions do not reveal the whole story, as, for example,

$$f: x \rightarrow \begin{cases} 1 & \text{if } x \text{ rational,} \\ 0 & \text{if } x \text{ irrational.} \end{cases}$$

Since high school students are not normally exposed to such functions, however, this is not a very serious obstacle.

The graph might best be introduced by using some function whose behavior is not too obvious. You may, for example, wish to use the "greatest integer contained in" function, which is easily explained and leads to some interesting configurations. We define

$$f:\ x \to [x]$$

as the function which maps x into the greatest integer contained in x. Thus

$$f(1) = 1,\quad f(\tfrac{3}{2}) = 1,\quad f(\tfrac{1}{2}) = 0,\quad f(-\tfrac{3}{2}) = -2, \text{ etc.}$$

The graph of the equation $y = [x]$ is in Figure TC. 1. There are a number of interesting combinations which can be formed with $[x]$.

Figures TC. 2 to TC. 4 illustrate three of them.

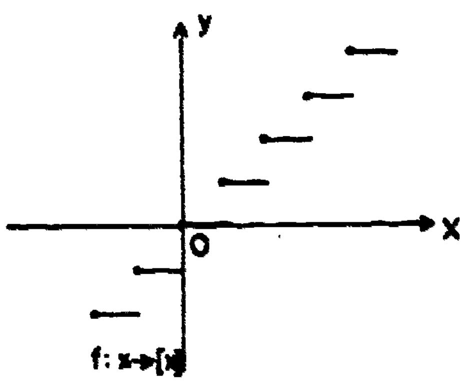


Figure TC. 1

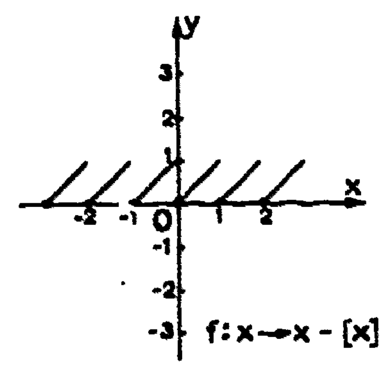


Figure TC. 2

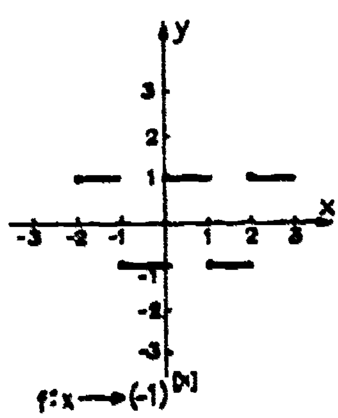


Figure TC. 3

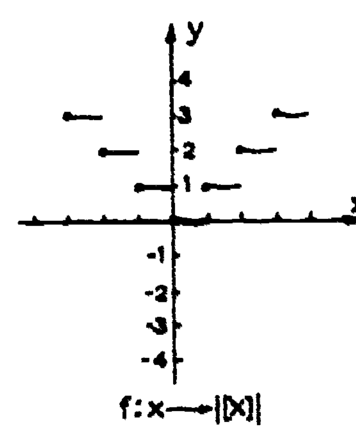


Figure TC. 4

Furthermore, an infinite checkerboard pattern is given by

$$\{(x,y):\ [x] \text{ is even and } [y] \text{ is even}\}.$$

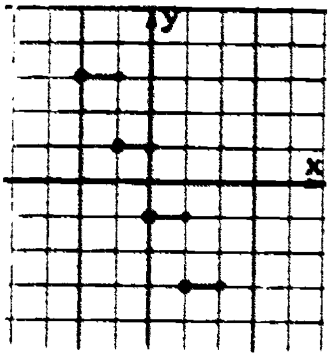
You may also find it helpful to sketch on the blackboard some figures similar to those in Figure 12 and to have the students determine whether or not they represent functions by applying the vertical line test. Problem 2 on page 9 is also a useful type of blackboard exercise, and you will probably find it helpful to do one as an illustration before the students attempt to do Exercises 2 themselves.

You may want to point out that most of the graphs in the text are incomplete.

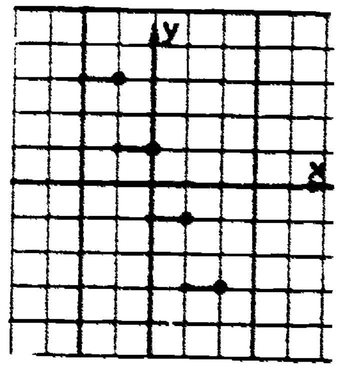
<u>Answers to Exercises 3</u>

1. (a) and (b)

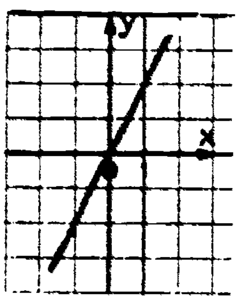
2. (a)



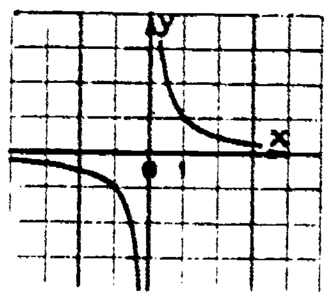
(b)



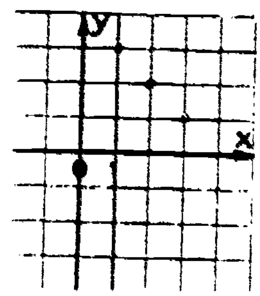
3. (a)



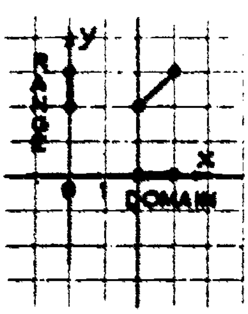
(b)



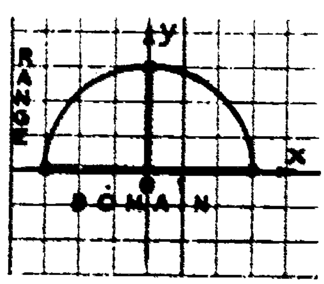
(c)



4. (a)



(b)



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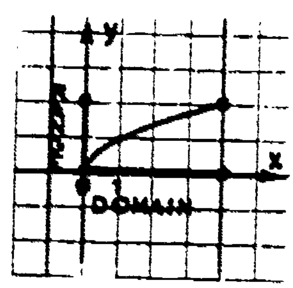

4. Constant Functions and Linear Functions.

Although the ideas of this section should be familiar to the student, it is unlikely that he will have encountered them in the language of mapping. This section will, therefore, give a review of important material and, at the same time, valuable practice in the use of functional notation.

Answers to Exercises 4

1. (a) slope $= 3$ (b) slope $= -2$ (c) slope $= -\frac{1}{2}$ (d) slope $= \frac{4}{3}$

2. (a) $f: x \to -2x + 6$ (b) $f: x \to -2x - 7$ (c) $f(x) = -2x + 7$ (d) $f(x) = -2x + 13$

3. (a) slope $= \frac{-3 - 4}{1 - 0} = -7$ (b) slope $= \frac{-3 - 3}{1 - 2} = 6$ (c) slope $= \frac{-3 - 5}{1 - 5} = 2$ (d) slope $= \frac{-3 + 13}{1 - 6} = -2$

4. (a) $f(x) = 3x - 2$ (b) $f(x) = -2x - 10$ (c) not a function (d) $f: x \to 4$

5. (a) $f: x \to -3x + 7$ (b) $f: x \to -3x - 3$ (c) $f(x) = -3x + 8$ (d) $f(x) = -3x - 13$

6. (a) $f(3) = 5$ (b) $f(3) = -3$ (c) $f(3) = 4$

7. Yes. The slope of the line through P and Q is -2, and the slope of the line through P and S is -2. Two lines through the same point having the same slope coincide.

8. (a) $(100.1 - 100)(\frac{39 - 25}{101 - 100}) + 25 = .1(14) + 25 = 26.4$

   $f(100.1) = 26.4$

   (b) $.3(14) + 25 = 4.2 + 25 = 29.2$

   $f(100.3) = 29.2$

   (c) $f(101.7) = 48.8$

   (d) $f(99.7) = 20.8$

9. (a) $f(53.3) = -44(.3) + 25 = 11.8$

   (b) $f(53.8) = -10.2$

   (c) $f(54.4) = -36.6$

   (d) $f(52.6) = 42.6$

10. $\begin{cases} 2x + 7y + 1 = 0 \\ x - 2y + 8 = 0 \end{cases}$

$\begin{cases} 2x + 7y + 1 = 0 \\ 2x - 4y + 16 = 0 \end{cases}$

$11y = 15$

$\begin{cases} y = \frac{15}{11} \\ x = -\frac{58}{11} \end{cases}$

$x - 3y + 4 = 0$

$\text{slope} = \frac{1}{3}$

$y = \frac{1}{3}x + b$

$\frac{15}{11} = \frac{1}{3}(-\frac{58}{11}) + b$

$\frac{103}{33} = b$

$y = \frac{x}{3} + \frac{103}{33}$ or

$11x - 33y + 103 = 0$

11. The slopes of the lines AB and CD are $\frac{1}{4}$, and the slopes of the lines AD and BC are $-\frac{3}{2}$. Since the opposite sides are parallel (have the same slope), ABCD is a parallelogram.

12. (a) $C(4,8)$ (b) $C(5,-11)$

13. $f\colon x \to 2x - 1$

$f(t + 1) = 2(t + 1) - 1 = 2t + 1$

$\therefore P(t + 1, 2t + 1)$ is on the graph of f.

14. $f(0) = f(t - 1)$ when $t = 1$. Then $f(0) = 3 \cdot 1 + 1 = 4$.

$f(8) = f(t - 1)$ when $t = 9$. Then $f(8) = 3 \cdot 9 + 1 = 28$.

15. $f(0) = f(t - 1)$ when $t = 1$. Then $f(0) = 1^2 + 1 = 2$.

$f(8) = f(t - 1)$ when $t = 9$. Then $f(8) = 9^2 + 1 = 82$.

16. $f(x_1) = mx_1 + b$, $f(x_2) = mx_2 + b$.

$$\begin{aligned} f(x_1) - f(x_2) &= mx_1 + b - (mx_2 + b) \\ &= mx_1 - mx_2 \\ &= m(x_1 - x_2). \end{aligned}$$

Since $m < 0$ and $x_1 < x_2$ or $x_1 - x_2 < 0$, $m(x_1 - x_2) > 0$.

$\therefore f(x_1) - f(x_2) > 0$ or $f(x_1) > f(x_2)$.

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5. The Absolute-Value Function.

There are many reasons for studying this function. It is simple but interesting and useful, and most high school students are unfamiliar with it. It is an important tool in many proofs and is used extensively in more advanced mathematics.

The definition $|x| = \sqrt{x^2}$ is an example of the composition of functions, considered at greater length in the next section; in this case, if $f\colon x \to x^2$

and g: $x \to \sqrt{x}$, then the absolute-value function is the compound function gf. You may want to mention this, informally, in anticipation of Section 6.

You may have to spend a little time on the definition of the square-root symbol, $\sqrt{}$. The definition is, of course, to a considerable extent arbitrary, but it must be unambiguous if it is to be useful. It would be a great inconvenience if $\sqrt{3}$, for example, represented a number which might be either positive or negative, and, to avoid this inconvenience, we agree that it is positive. We can then represent the negative number whose square is 3, without ambiguity, as $-\sqrt{3}$. Because, for example, $\sqrt{6^2} = 6$, students find it tempting to write $\sqrt{x^2} = x$. This is, of course, false if $x < 0$. The only correct statement that can be made here is, in fact, the second definition of absolute value; Exercise 1 is designed to reinforce this.

It is convenient, in many applications, to think of $|x|$ as the (undirected) distance on the number line between the origin and the point x. Similarly, $|x - a|$ (or $|a - x|$) is the distance between the point x and the point a. This concept is particularly convenient in problems like Exercises 3 and 4; thus, for example, the values of x which satisfy $|x - 5| < 2$ are those values which are less than 2 units from 5, namely, all those from 3 to 7. Inequalities such as these appear fairly often in the calculus.

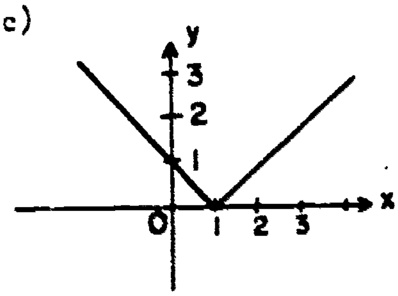
Exercises 1-8 are intended to give the pupil an understanding of the meaning of absolute value and some facility in manipulating it. Exercises 9-11 are in anticipation of further work on inequalities.

Answers to Exercises 5

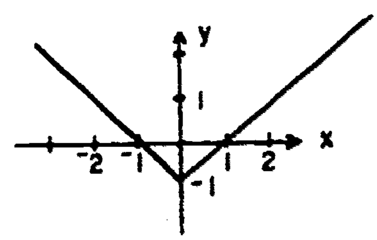
1. (a) $x \geq 0$ (b) $x \leq 0$. These follow directly from the definition of the square-root symbol, $\sqrt{}$.

2. (a) $x - 1 \geq 0$ or $x \geq 1$ (b) $x - 1 \leq 0$ or $x \leq 1$

(c)



(d)



3. (a) The points on the number line 14 units from 0 are 14 and -14. A more formal (and longer) way to arrive at the same result is to note that either $x \geq 0$, in which case $|x| = x$ and the given equation then reads $x = 14$, or $x < 0$, in which case $|x| = -x$ and the equation reads $-x = 14$ or $x = -14$.

Still another approach:

$$|x| = \sqrt{x^2} = 14$$
$$x^2 = 196$$
$$x = \pm 14 \quad \text{(and both check).}$$

(b) $x = 5$ or -9. All three methods discussed under (a) apply here, with $x + 2$ replacing x.

(c) Since the absolute value of a number is never negative, it should be clear by inspection that the equation has no roots.

4. (a) The problem asks for those points on the number line which are within 1 unit of 2. These are the numbers from 1 to 3, and the solution is therefore $\{x:\ 1 < x < 3\}$. It is a common practice merely to give the double inequality which defines this set, and state the solution as $1 < x < 3$.

(b) Here we must find those points which are <u>more</u> than 2 units from 5; hence $\{x:\ x < 3 \text{ or } x > 7\}$.

(c) $\{x:\ -4.2 < x < -3.8\}$, as in (a). Note that $|x + 4| = |x - (-4)|$ is the distance between x and -4.

(d) By Theorem 1, $|2x - 3| = 2|x - 1.5|$; hence, the given inequality becomes

$$2|x - 1.5| < 0.04$$
$$|x - 1.5| < 0.02,$$

and the solution, as in (a), is $\{x:\ 1.48 < x < 1.52\}$.

(e) $\{x:\ -1.28 < x < -1.22\}$, as in (d).

5. If $x \geq 0$, then $|x| = x$ and $x \cdot |x| = x^2$; if $x < 0$, then $x \cdot |x| = x(-x) = -x^2 < 0 < x^2$.

6. In Theorem 2, replace b by $-b$; since $|-b| = |b|$, the desired result follows immediately.

7. If $a > b$, then $|a - b| = a - b$ and the given expression becomes

$$\frac{1}{2}(a + b + a - b) = a.$$

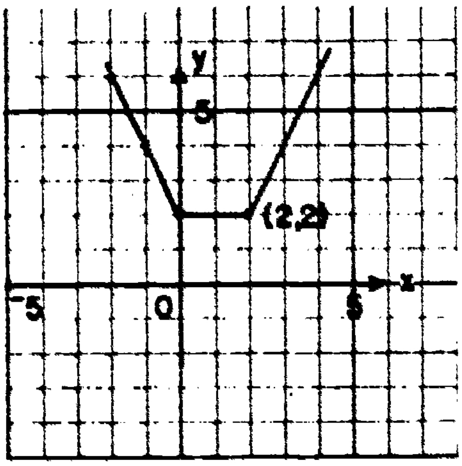
If $a < b$, then $|a - b| = -a + b$ and the given expression becomes

$$\frac{1}{2}(a + b - a + b) = b.$$

The lesser of a and b is given by $\frac{1}{2}(a + b - |a - b|)$. Trial is as good a way as any to get this.

8. If $x < 0$, then $|x| = -x$, $|x - 2| = -x + 2$, and $y = -2x + 2$.
If $0 \leq x < 2$, then $|x| = x$, $|x - 2| = -x + 2$, and $y = 2$.
If $x \geq 2$, then $|x| = x$, $|x - 2| = x - 2$, and $y = 2x - 2$.

Hence we get



9. $|x^2 + 2x| \leq |x^2| + |2x|$ by Theorem 2

$\leq |x^2| + 2|x|$ by Theorem 1

$< |x| + 2|x| = 3|x|$ by the inequality given in the Exercise.

10. Multiplying both sides of $x < k$ by the positive number x gives $x^2 < kx$. Then, proceeding as in Exercise 9, we get $|x^2 - 3x| \leq |x^2| + |-3x| = |x^2| + 3|x| < 0.1|x| + 3|x| = 3.1|x|$ if $|x| < 0.1$.

11. $|x| < 0.001$, or $-0.001 < x < 0.001$. This result can be established as in Exercise 10.

Answers to Exercises 6

1. (a) -1 (b) 1 (c) 5 (d) 63 (e) $x^2 + 4x + 3$ (f) $x^2 + 1$ (g) $x + 5$

2. (a) $acx + ad + b$ (b) $acx + cb + d$

(d) Theorem: The slope of either composite of two linear functions is equal to the product of the slopes of the two linear functions.

3. (a) 1, -3, 8

(b) If $f: x \to \frac{1}{x}$, then $ff: x \to x$ for all $x \neq 0$.

4. (a) $fj: x \to x + 2$ $jf: x \to x + 2$

(b) $g: x \to x - 2$

(c) $h: x \to x - 2$

5. (a) $(fg)(x) = x^6$ and $(gf)(x) = x^6$

(b) $(fg)(x) = (gf)(x) = x^{mn}$

6. (a) $(f \cdot g)(x) = x^5$ (b) $(f \cdot g)(x) = x^{m+n}$

7. (a) $(f \cdot g)(x) = x^2 - x - 6$
(b) $\big((f \cdot g)h\big)(x) = x^4 - x^2 - 6$
(c) $(fh)(x) = x^2 + 2$
(d) $(gh)(x) = x^2 - 3$
(e) $\big((fh) \cdot (gh)\big)(x) = x^4 - x^2 - 6$

8. The result is true and can be proved as follows: Given three functions, $f\colon\ x \to f(x)$, $g\colon\ x \to g(x)$, and $h\colon\ x \to h(x)$, we wish to show that

$$(f \cdot g)h = (fh) \cdot (gh).$$

(It is assumed throughout that f, g, and h are being discussed for all x in the intersection of their domains.)

By definition: $(f \cdot g)(x) = f(x) \cdot g(x)$,

hence $$\big((f \cdot g)h\big)(x) = (f \cdot g)\big(h(x)\big) = f\big(h(x)\big) \cdot g\big(h(x)\big) = \big((fh) \cdot (gh)\big)(x).$$

9. The theorem is false, as the following counter-example shows: Take for both g and h the identity function $x \to x$ and for f the function $x \to x + 1$. Then $(g \cdot h)(x) = x^2$ and $\big(f(g \cdot h)\big)(x) = x^2 + 1$, but $(fg)(x) = (fh)(x) = x + 1$, and therefore $\big((fg) \cdot (fh)\big)(x) = (x + 1)^2 \neq x^2 + 1$.

10. $(f + g)h = fh + gh$, since $(f + g)(x) = f(x) + g(x)$, and if x is replaced by $h(x)$, we obtain $(f + g)\big(h(x)\big) = f\big(h(x)\big) + g\big(h(x)\big)$. But $f(g + h) \neq fg + fh$; this can be shown using as counter-example the functions suggested under Exercise 9.

11. Take $f(x) = m_1x + b_1$, $g(x) = m_2x + b_2$, $h(x) = m_3x + b_3$.

Then $(gh)(x) = g\big(h(x)\big) = m_2(m_3x + b_3) + b_2 = m_2m_3x + m_2b_3 + b_2$,

and $$\big(f(gh)\big)(x) = f\big((gh)(x)\big) = m_1(m_2m_3x + m_2b_3 + b_2) + b_1 = m_1m_2m_3x + m_1m_2b_3 + m_1b_2 + b_1.$$

Similarly, $$(fg)(x) = f\big(g(x)\big) = m_1(m_2x + b_2) + b_1 = m_1m_2x + m_1b_2 + b_1$$

and $$\big((fg)h\big)(x) = (fg)\big(h(x)\big) = m_1m_2(m_3x + b_3) + m_1b_2 + b_1 = m_1m_2m_3x + m_1m_2b_3 + m_1b_2 + b_1 = \big(f(gh)\big)(x).$$

Since this result is valid for all $x \in R$, it follows that

$$(fg)h = f(gh).$$

---

7. Inversion.

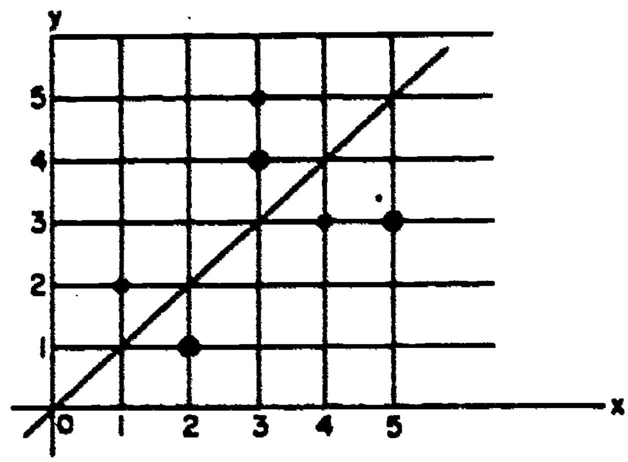
It is frequently helpful in conveying the idea of an inverse to consider functions with a finite domain. Thus, let $f: x \to y$ be described by the table

x	1	3	4
y	2	5	3

Domain of $f = \{1, 3, 4\}$

Range of $f = \{2, 5, 3\}$

which we may represent on a graph as dots.



f^{-1} undoes what f does. Thus, if f sends 1 into 2, f^{-1} sends 2 right back to 1. Hence, the domain of f^{-1} is the range of f, $\{2, 5, 3\}$, and the range of f^{-1} is the domain of f. We may write the table for f^{-1} as

x	2	5	3
y	1	3	4

pictured on the above graph as circles. Note the symmetry of f and f^{-1} with respect to the graph of $y = x$.

When the expression on the right side of the arrow is simple, f^{-1} may be easily obtained from f. Thus, if $f: x \to 3x - 2$, then $f^{-1}: x \to \frac{x + 2}{3}$. f corresponds to the instructions "multiply by 3 and then subtract 2." To "undo" this and obtain f^{-1}, we add 2 and then divide by 3. This is all well and good for simple functions. However, this approach no longer works if $f: x \to \frac{x + 1}{x + 2}$.

If we write $f: x \to y$ where $y = \frac{x + 1}{x + 2}$, when we seek f^{-1} we want to find the value of x that is associated with a particular y. Hence, we solve $y = \frac{x + 1}{x + 2}$ for x, obtaining $x = \frac{2y - 1}{-y + 1}$. We then usually write $f^{-1}: x \to \frac{2x - 1}{-x + 1}$ using x in place of y.

We could check this result quickly by taking a specific value for x, say $x = 0$, and seeing whether f^{-1} undoes what f does. Thus,

f: $0 \to \frac{0+1}{0+2} = \frac{1}{2}$ or $f(0) = \frac{1}{2}$. f^{-1} should send $\frac{1}{2}$ back to 0. Let us see if it does. f^{-1}: $\frac{1}{2} \to \frac{2(\frac{1}{2}) - 1}{-(\frac{1}{2}) + 1} = \frac{1-1}{\frac{1}{2}} = 0$ and it does. In general we have

$$(f^{-1}f)(x) = f^{-1}\left(f(x)\right) = f^{-1}\left(\frac{x+1}{x+2}\right)$$

$$= \frac{2\frac{x+1}{x+2} - 1}{-\frac{x+1}{x+2} + 1} = \frac{2(x+1) - (x+2)}{-(x+1) + (x+2)}$$

$$= \frac{2x + 2 - x - 2}{-x - 1 + x + 2} = \frac{x}{1}$$

$$= x.$$

Moreover, $(ff^{-1})(x) = f\left(f^{-1}(x)\right) = f\left(\frac{2x-1}{-x+1}\right)$ or

$$(ff^{-1})(x) = \frac{\frac{2x-1}{-x+1} + 1}{\frac{2x-1}{-x+1} + 2} = \frac{2x - 1 + (-x+1)}{2x - 1 + 2(-x+1)} = \frac{x}{1} = x.$$

In general, if f: $x \to y = f(x)$, then solve the equation $y = f(x)$ for x in terms of y. This enables us to associate with a given y its x-partner and thus reveals the inverse, f^{-1}. Of course, if f does not have an inverse, the expression obtained for x in terms of y will reveal this.

Answers to Exercises 7

1. (a) $x \to x + 7$ (b) $x \to \frac{x-9}{5}$ (c) $x \to \frac{1}{x}$

2. (a) $x = y + 7$ compared with $x \to x + 7$

   (b) $x = \frac{y-9}{5}$ compared with $x \to \frac{x-9}{5}$

   (c) $x = \frac{1}{y}$ compared with $x \to \frac{1}{x}$

3. Let the number be x; then the various instructions given can be represented by the functions f_1 to f_7, as follows:

$$f_1: x \to 5x$$

$$f_2: x \to x + 6$$

$$f_3: x \to 4x$$

$$f_4: x \to x + 9$$

$$f_5: x \to 5x$$

$$f_6: x \to x - 165$$

$$f_7: x \to \frac{x}{100}$$

Then $f_7f_6f_5f_4f_3f_2f_1$: $x \to \frac{5\{4(5x + 6) + 9\} - 165}{100} = x.$

<u>Answers to Exercises 8</u>

1. (a) $x \to \frac{x + 5}{4}$ (b) $x \to \frac{3}{x - 8}$ (c) $x \to \sqrt[3]{x + 2}$

2. (a) $x = \frac{y + 5}{4}$ (b) $x = \frac{3}{y - 8}$ (c) $x = \sqrt[3]{y + 2}$

3. Suppose the digits are x and y, and we pick x. Then we define:

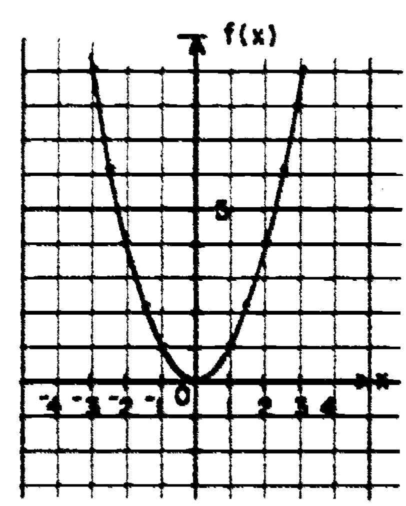
$$f_1: \ x \to 5x$$
$$f_2: \ x \to x + 7$$
$$f_3: \ x \to 2x$$
$$f_4: \ x \to x + y$$
$$f_5: \ x \to x - 14$$
$$f_5f_4f_3f_2f_1: \ x \to 2(5x + 7) + y - 14 = 10x + y,$$

a number with tens digit x and units digit y.

4. A function which has an inverse.

5. If $x_1 \neq x_2$, then either $x_1 < x_2$, in which case $f(x_1) > f(x_2)$, or $x_1 > x_2$, in which case $f(x_1) < f(x_2)$. In either case, $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$; hence f is one-to-one and has an inverse by Theorem 4.

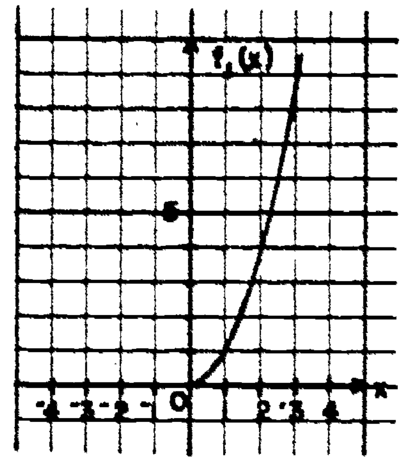
6. (a) $f(1) = 1 = f(-1)$ suffices to show that f is not one-to-one and therefore does not have an inverse.



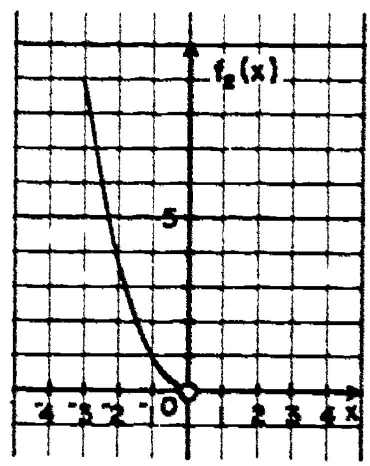
$f: \ x \to x^2$

(b) f_1^{-1}: $x \to \sqrt{x}$

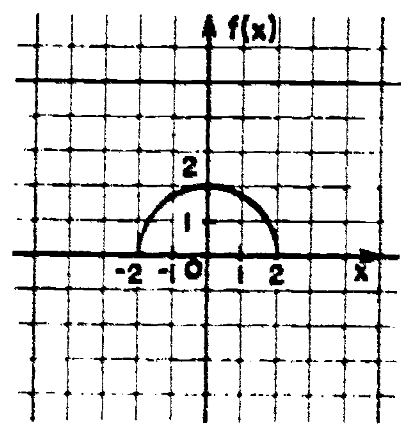
f_2^{-1}: $x \to \sqrt{-x}$



(c) The domain of f is the union of the domains of f_1 and f_2.



7. (a) $f(1) = \sqrt{3} = f(-1)$ suffices to show that f is not one-to-one and therefore does not have an inverse.



f: $x \to \sqrt{4 - x^2}$

(b) For example, f_1: $x \to \sqrt{4 - x^2}$, $0 \leq x \leq 2$, and

f_2: $x \to \sqrt{4 - x^2}$, $-2 \leq x \leq 0$.

8. f_1: $x \to x^2 - 4x$, $x \geq 2$, and f_2: $x \to x^2 - 4x$, $x < 2$.

9. All $x \to x^3 - 3x$, with domains $\{x: x < -1\}$, $\{x: -1 \leq x \leq 1\}$, and $\{x: x > 1\}$.

Answers to Exercises 9

1. (a) $\{(0,-1), (2,5), (5,14)\}$ (c) $\{(x,2): x \text{ an integer}\}$
   (b) $\{(x,x^3): x \in R\}$ (d) $\{(x,x): x \in R\}$

2. (a) $f: 0 \to 1, 2 \to 3, 4 \to 5$, or $f: x \to x + 1, x \in \{0, 2, 4\}$
   (b) $f: x \to \sqrt{x}$, x a positive real number
   (c) $f: x \to -1, x \in R$
   (d) $f: 0 \to -2, -1 \to 4, 5 \to 15$

3. (a), (c), and (d) do; (b) does not since both $(2,3)$ and $(2,5)$ are in the set.

4. (c) and (d) do; (a) does not since $(5,1)$ and $(6,1)$ are both in the set. The inverse of (c) is $\{(1,-1), (-2,3), (0,0)\}$; the inverse of (d) is $\{(2,-1)\}$.

5. (a) $\{(1,0), (3,2), (5,4)\}$
   (b) $\{(x,x^2): x$ a positive real number$\}$
   (c) Does not have an inverse since, for example, $1 \to -1$ and also $2 \to -1$.
   (d) $\{(-2,0), (4,-1), (15,5)\}$

Answers to Miscellaneous Exercises

1. If we write g as $\{(x,g(x)): x \in \text{domain of } g\}$ and $f = \{(x,f(x)): x \in \text{domain of } f = \text{range of } g\}$, then $fg = \{(x,fg(x)): x \in \text{range of } g\}$.

2. (a) function (d) not a function
   (b) function (e) not a function
   (c) function with inverse (f) function

3. $f: x \to 2$ or $y = 2$.

4. When $a = b = 0$ and $c \in R$.

5. The point of intersection is $(\frac{1}{5}, -\frac{7}{5})$, so $f: x \to -\frac{7}{5}$.

6. The point of intersection is $(\frac{b - 4}{a - 5}, \frac{ab - 20}{a - 5})$, provided $a \neq 5$. If $a = 5$, they will be parallel, or if also $b = 4$, they will coincide.

7. $f_1: x \to x + 3$ and $f_2: x \to -x - 3$.

8. $10x + y - 7 = 0$ or $y = 7 - 10x$ and $m = -10$. This means for each unit increase in x, y <u>decreases</u> by 10. If x increases from 500 to 505, y decreases 50. If y decreases from -500 to -505, x increases $.5$ or $\frac{1}{2}$.

9. Slope of line $= \frac{3-1}{2-(-1)} = \frac{2}{3}$ and line is $y = \frac{2}{3}x$.

10. Point of intersection is $(0,k)$, so line is $y = \frac{5}{6}x + k$.

11. $6x + 3y - 7 = 0 \longrightarrow y = \frac{7}{3} - 2x$

$y = -2x + 3 \longrightarrow y = 3 - 2x$

Lines are parallel and $\frac{2}{3}$ of a unit apart on the y-axis. Half of this distance is $\frac{1}{3}$ so the line is $y = \frac{8}{3} - 2x$.

12. $2y = x + 3 \longrightarrow y = \frac{x}{2} + \frac{3}{2}$. Slope $= \frac{1}{2}$.

Slope of $\perp$ is -2. So line is $y = -2x + 18$.

13. $y = f_1(t) = t - 10$ when t is in minutes.
$y = f_2(t) = 60(t - \frac{1}{6})$ when t is in hours. Domain of t for f_1 is $\{t:\ t \geq 10$ and t is a natural number$\}$; for f_2 it is $\{t:\ t \geq \frac{n}{60}$ and $n \geq 10$ and n is a natural number$\}$.

14. (a) AC: $y = \frac{7}{12}x$
(b) BD: $y = 14 - \frac{7}{4}x$
(c) Intersection is point $(6, 3\frac{1}{2})$

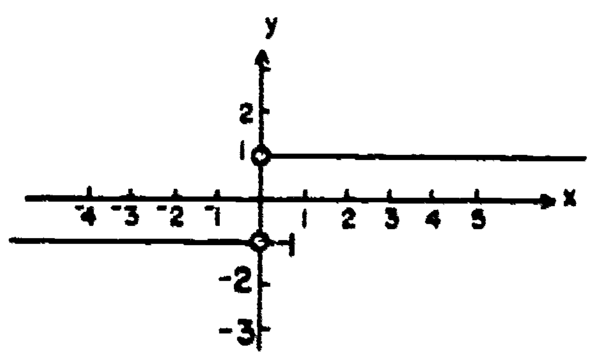
15. (a) AC: $y = \frac{y_2}{x_2}x$
(b) BD: $y = \frac{y_2}{x_2 - 2x_1}(x - x_1)$
(c) Intersection is point $(\frac{x_2}{2}, \frac{y_2}{2})$

16. $(f(gh))(x) = f(g(h(x))) = f(g(c)) = f(b) = a$,
$(f(hg))(x) = f(h(g(x))) = f(h(b)) = f(c) = a$.
But $gh:\ x \to b$ and $hg:\ x \to c$, so these two are different unless $b \neq c$.

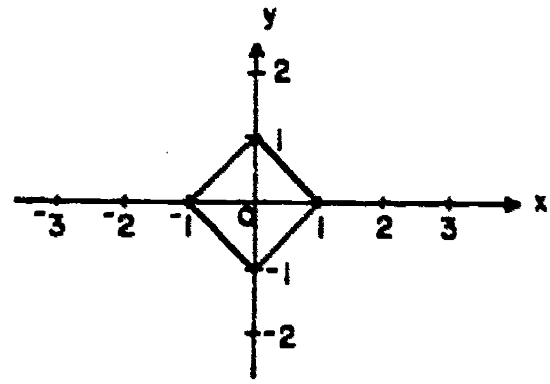
17. $f^{-1}:\ x \to \frac{x - b}{m}$.

18. Any constant function $x \to c$, or the identity function $x \to x$, or the absolute-value function $x \to |x|$.

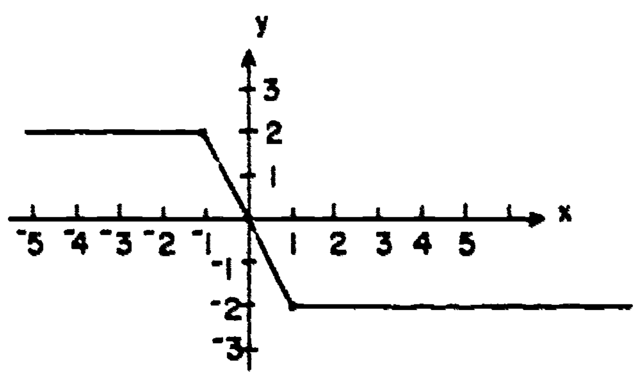
19. (a)



(b) Note that neither absolute value can exceed 1; hence x is limited to $\{x: -1 \le x \le 1\}$ and y is similarly restricted.



(c)



20. $(fg)(x) = 2(3x + k) - 5 = 6x + 2k - 5$

$(gf)(x) = 3(2x - 5) + k = 6x - 15 + k$

$6x + 2k - 5 = 6x - 15 + k$

$k = -10$

21.

	Domain	Range
f	R	$\{y: y \ge 0\}$
g	$\{x: \lvert x\rvert \le 4\}$	$\{y: 0 \le y \le 4\}$

The intersection of the range of f and the domain of g is $\{y: 0 \le y \le 4\}$. The elements of this set are the images, under the mapping f, of $\{x: |x| \le 2\}$, which is therefore the domain of gf.

The intersection of the range of g and the domain of f is the range of g itself; hence the domain of fg is the domain of g, that is, $\{x: |x| \le 4\}$.

Illustrative Test Questions

The following questions (with answers appended) have been included in order to assist you in the preparation of tests and quizzes. The order of the items is approximately the same as the order in which the various concepts being tested appear in the text. This means that you can use selected problems from this list before the chapter has been completed.

For a short quiz, one or two problems from this list would be sufficient; a full period (40 or 50 minutes) test might contain anywhere from five to ten of the problems. It would be a mistake to give all the questions as a chapter test unless at least two class periods were planned for it.

1. Given $f: x \to x^2 + 2$, find

   (a) $f(3)$; (b) $f(6)$; (c) $f(\frac{3}{2})$.

2. Find the domain of f if it is the largest set of real numbers that f maps into real numbers, and find also the corresponding range:

   (a) $f: x \to \sqrt{x - 1}$

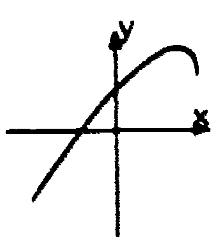
   (b) $f: x \to \dfrac{x + 2}{x + 1}$

3. Which of these could be the graph of

   (a) a function $f: x \to y$?

   (b) a function $f: y \to x$?

(1)



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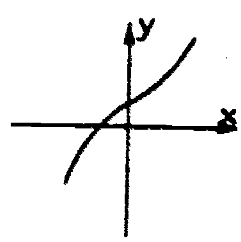

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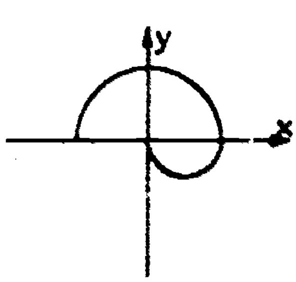

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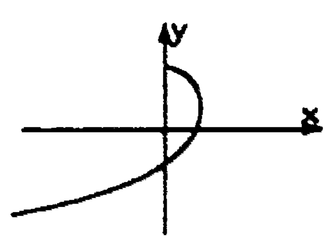


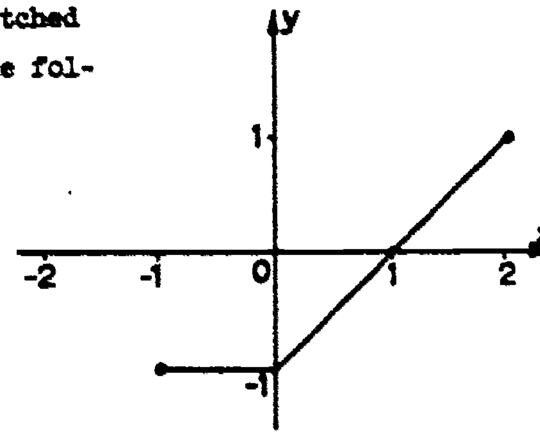
4. Given the function $f: x \to f(x)$ sketched at the right, sketch the graphs of the following functions:

(a) $g: x \to -f(x)$

(b) $h: x \to -f(-x)$

(c) $k: x \to f(|x|)$

(d) $m: x \to |f(x)|$



5. Graph the following functions, indicating the domain and range on the appropriate axes:

(a) $f: x \to 1 + \sqrt{x}, \quad x > 1$

(b) $g: x \to 1 + |x - 1|, \quad -1 \leq x \leq 2$

6. Graph: $|x| + 2|y| = 4$.

7. Solve: (a) $|x + 3| < 0.2$;

(b) $|2x - 5| < 0.1$.

8. Find a linear function f such that $f(2) = 3$ and $f(3) = 2f(4)$.

9. If a linear function f has slope $\frac{3}{2}$ and if $f(2) = -3$, find $f(7)$.

10. What is the slope of a linear function f if $f(5) - f(2) = 4$?

11. Find the linear function whose graph passes through all points with coordinates of the form $\big((t + 3)(t - 2), (t + 4)(t - 3)\big)$.

12. Find the value of k for which $(k, 2k)$ lies on the line through $(3,-2)$ and $(5,4)$.

13. Given $f: x \to 3x + 1$ and $g: x \to x^2 - 2$, find the function $fg - gf$.

14. Given $f: x \to 2x + 1$ and $g: x \to x^2 - 1$, solve the equation $(gf)(x) = 0$.

15. Given $f: x \to 3x + 5$ and $g: x \to 2x + k$, find k if $(gf)(x) = (fg)(x)$ for all $x \in R$.

16. Write the following functions, defined as mappings, as sets of ordered pairs:

(a) $f: 1 \to 2, -1 \to 0$

(b) $f: x \to 5x, x \in R$

17. Write the following functions defined as sets of ordered pairs as mappings:

(a) $\{(0,1), (3,-5), (2,7)\}$

(b) $\{(x, 2x - 1): x \in R\}$

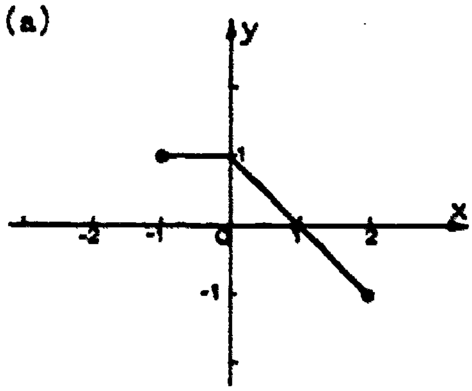
Answers to Illustrative Test Questions

1. (a) 11 (b) 38 (c) $\frac{17}{4}$

2. (a) Domain: $\{x: x \geq 1\}$. Range: $\{y: y \geq 0\}$.
   (b) Domain: $\{x: x \neq -1\}$. Range: $\{y: y \neq 1\}$.

3. (a) (1), (2) (b) (2), (4)

4. (a)



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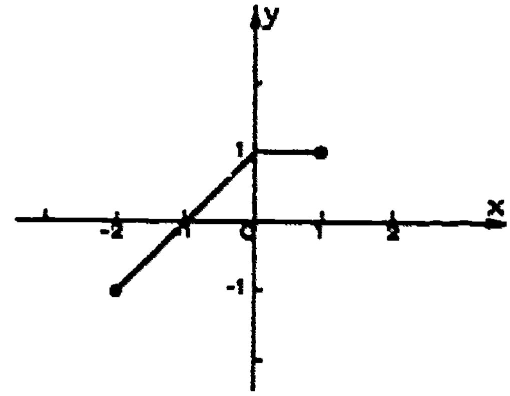

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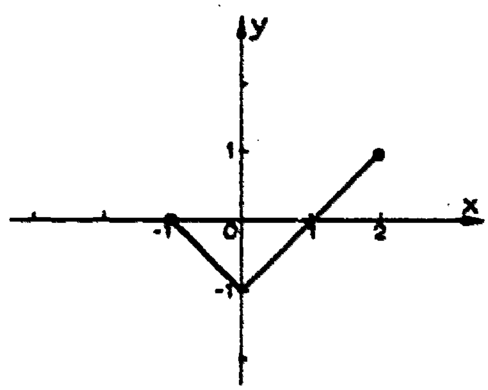

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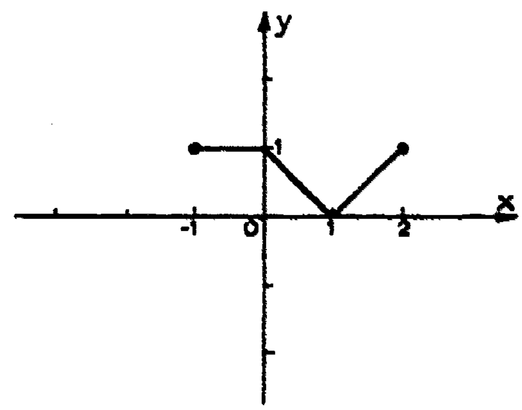
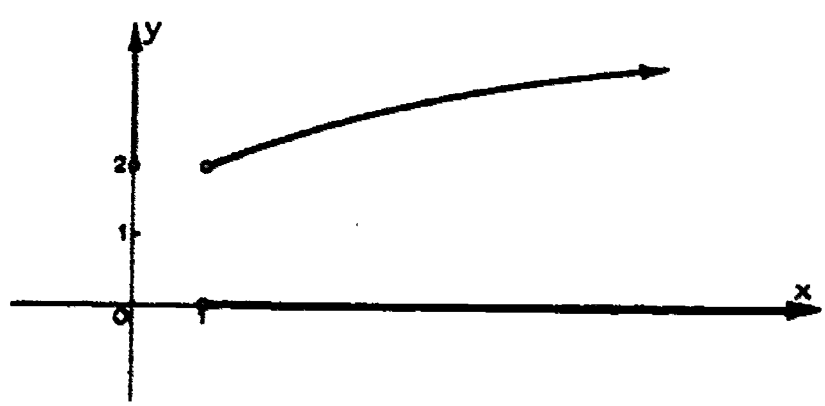


5. (a)



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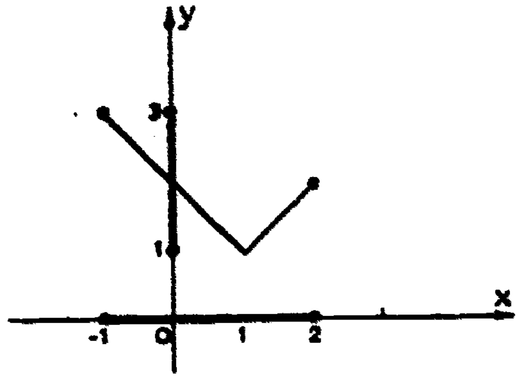
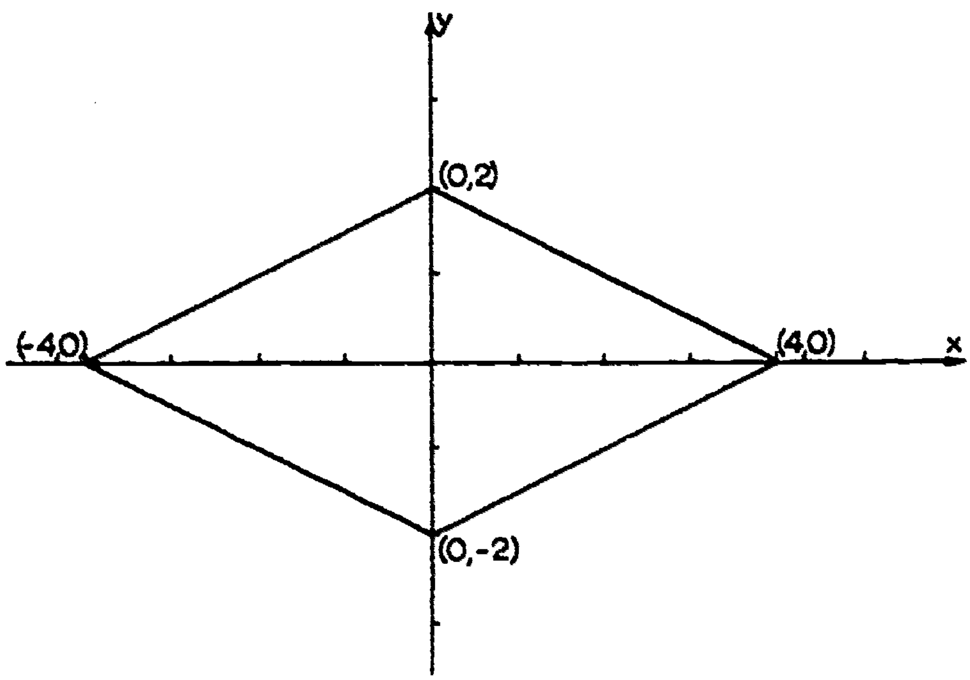


6.



7. (a) $-3.2 < x < -2.8$ (b) $2.45 < x < 2.55$

8. $x \rightarrow -x + 5$

9. $f(7) = \frac{9}{2}$

10. $m = \frac{4}{3}$

11. $x \rightarrow x - 6$

12. $k = 11$

13. $fg - gf\colon\ x \rightarrow -6x^2 - 6x - 4$

14. $x = 0, -1$

15. $k = \frac{5}{2}$

16. (a) $\{(1,2), (-1,0)\}$ (b) $\{(x,5x)\colon\ x \in R\}$

17. (a) $f\colon\ 0 \rightarrow 1,\ 3 \rightarrow -5,\ 2 \rightarrow 7$ (b) $f\colon\ x \rightarrow 2x - 1,\ x \in R.$

CIRCULAR FUNCTIONS

Introduction.

The pamphlet is not a course in trigonometry in the solution-of-triangles sense. It is expected that, normally, this aspect of trigonometry will have been studied previous to the present chapter. (This is not unconditionally necessary, however.)

Emphasis is placed on the periodic property of sin and cos. (A relatively small part of the pamphlet is devoted to tan.) Extensive use is made of the idea of rotating the plane about a perpendicular to it through the origin. This gives a certain unity to the discussion. Consistent with this emphasis, we have derived the formulas for $\sin(x + y)$ and $\cos(x + y)$ in terms of the simplest properties of rotation. We believe that this approach is a natural one (no tricks!) and that the student will really understand analytic trigonometry when he studies it in this way.

1. Circular Motions and Periodicity.

The emphasis throughout is on the periodic properties of the circular functions, i.e., the sine and cosine. In beginning, you should emphasize that we shall talk here about functions which differ from those we have previously studied in that they have the property of periodicity.

One good way to visualize a periodic function is in terms of the machine developed in the pamphlet Functions. If the function depicted by the machine is periodic of period a, then when $x, x + a, x + 2a, \ldots, x + na$ are dropped into the hopper, we obtain the same output, $f(x)$, in each case. In the next section we speak of laying rectangles containing one complete cycle of the function end to end, and you may wish to use the idea here in order to illustrate further the meaning of periodicity.

The use of the uv- and xy-planes which we employ may be a source of difficulty at first. We wish to talk about the unit circle with which we define sin and cos, but later we shall need to display the graphs of $y = \sin x$ and $y = \cos x$ on an xy-plane. Since we are using x for arc length (to obtain the familiar $\sin x$ and $\cos x$) it might be confusing to teach the student to visualize x as both the horizontal axis on the plane of the unit circle and at the same time a length of circular arc. We feel that if care is exercised at the time the transition is made in Section 2, the use of u and v is more satisfactory than trying to get x to wear two hats in this section.

A more exact way of defining sin and cos is by a composition of two functions, one from the set of real numbers to the set of geometric points on the unit circle and the other from the set of points on the circle to the set of real numbers. Thus, if $x \in R$ and if P is a point on the unit circle, we have a function

$$f: x \to P$$

and another function

$$g: P \to \cos x$$

from which

$$gf: x \to \cos x$$

and similarly for the sine. We feel, however, that the way in which we have handled it in the text, while possibly less rigorous, is certainly easier to teach and is perfectly adequate for our purposes.

The fact that cos and sin are functions from real numbers to real numbers should be emphasized. You might point out to the student that nowhere is this section have we used an angle and, although we have used the concept of arc length, sine and cosine are completely divorced from any geometric considerations. They are functions on the set of real numbers in the same sense as polynomials, say, or exponential functions. Too often when we speak of $\sin A$, the students feel that A must be an angle. Sometimes they think of A as being the degree measure or radian measure of an angle, but the idea that A need have no connection with an angle is usually very strange.

Exercises 1

The exercises lean on the notion of periodicity. The first five are not difficult. We have starred Exercises 6-9 since they require more insight than the others, but if Problem 7a is not assigned as homework, it should be covered in class, since this relationship is used in Section 4.

Answers to Exercises 1

1. The rotation of the earth about the sun every $365\frac{1}{4}$ days.

   The phases of the moon; period is about $29\frac{1}{2}$ days.

   The swinging of the pendulum of a clock; for a grandfather's clock, the period is usually 2 seconds.

   The oscillation of a piston in a steam engine or internal combustion engine, period depends upon speed of engine.

   The alternation of A.C. electric current; for 60 cycle current, the period is $\frac{1}{60}$ second.

Oscillation of vacuum tubes, vibrations of strings of musical instruments (sound waves in general), etc.

2. (a) $p(-\frac{\pi}{2}) = p(\frac{3\pi}{2} - 2\pi) = p(\frac{3\pi}{2})$

(b) $p(3\pi) = p(\pi + 2\pi) = p(\pi)$

(c) $p(-\frac{3\pi}{2}) = p(\frac{\pi}{2} - 2\pi) = p(\frac{\pi}{2})$

(d) $p(4076\pi) = p(0 + 2038 \cdot 2\pi) = p(0)$

3. (a) $(0,-1)$ (c) $(0,1)$

(b) $(-1,0)$ (d) $(1,0)$

4. (a) $x = \frac{3\pi}{2}, \frac{7\pi}{2}$ (c) $x = 0, 2\pi$

(b) $x = \pi, 3\pi$ (d) $x = \pi, 3\pi$

5. (a) $x = \frac{\pi}{4}, \frac{5\pi}{4}$ (b) $x = \frac{3\pi}{4}, \frac{7\pi}{4}$

*6. (a) $\sin 2x = \sin (2x + 2\pi)$ from periodicity of sin

$= \sin 2(x + \pi)$, and the period is π.

(b) $\sin \frac{1}{2}x = \sin (\frac{1}{2}x + 2\pi)$

$= \sin \frac{1}{2}(x + 4\pi)$, and the period is 4π.

(c) $\cos 4x = \cos (4x + 2\pi)$

$= \cos 4(x + \frac{\pi}{2})$, and the period is $\frac{\pi}{2}$.

(d) $\cos \frac{1}{2}x = \cos (\frac{1}{2}x + 2\pi)$

$= \cos \frac{1}{2}(x + 4\pi)$, and the period is 4π.

*7. (a) $f(x) = f(x + a)$, $g(x) = g(x + a)$. Given.

$f(x) + g(x) = f(x + a) + g(x + a)$. Addition Axiom.

$(f + g)(x) = (f + g)(x + a)$. By definition.

$\therefore f + g$ is periodic with period a. By definition.

To show that a is not necessarily the fundamental period, you can use, for example,

$f: x \to \ln \sin x$ and $g: x \to \ln \cos x$,

each of which has period 2π. But

$$(f + g)(x) = \ln \sin x + \ln \cos x = \ln (\sin x \cos x) = \ln (\tfrac{1}{2} \sin 2x)$$

and $f + g$ therefore has fundamental period π.

An even more striking example is afforded by

$f: x \to \sin x$ and $g: x \to -\sin x$;

then $f + g: x \to 0$ and has every real number as a period, but has no fundamental period.

(b) $f(x) \cdot g(x) = f(x + a) \cdot g(x + a)$ Multiplication Axiom.

$(f \cdot g)(x) = (f \cdot g)(x + a)$ Definition.

$\therefore f \cdot g$ is periodic with period a. Definition.

An example in which a is not the fundamental period is

$$f: x \rightarrow \sin x \quad \text{and} \quad g: x \rightarrow \cos x$$

which yields

$$f \cdot g: x \rightarrow \sin x \cos x = \frac{1}{2} \sin 2x$$

with fundamental period π.

*8. $f(x) = f(x + a)$ Given.

$g(x) = g(x)$ if g is defined at x.

$g\big(f(x)\big) = g\big(f(x + a)\big)$ Substitution.

$(gf)(x) = (gf)(x + a)$ By definition of gf.

$\therefore gf$ is periodic with period a.

*9. If a is a period of cos, it must be true that

$$\cos (x + a) = \cos x$$

for all $x \in R$. In particular, it must be true if $x = 0$ that

$$\cos a = \cos 0 = 1.$$

But the only point on the unit circle with abscissa 1 is $(1,0)$, which corresponds to $x = 0 + 2n\pi$.

The proof for sin is similar; use $x = \frac{\pi}{2}$.

---

2. Graphs of Sine and Cosine.

The rectangle device used here can be a very useful one in teaching the student to graph periodic functions. By establishing the period and amplitude visually, it directs his attention to a specific region of the plane with respect to both the domain and range of the function.

We use the geometric argument to obtain specific values of the functions, because it is the simplest and most familiar tool available to the student. We hope that you will emphasize the symmetric nature of the unit circle and that the student will be encouraged to use considerations of symmetry whenever possible.

Exercises 2

The exercises develop some simple symmetry properties of $\sin x$ and $\cos x$, and lead the student into understanding the effect of the constants A, B, and C in $y = A \sin (Bx + C)$.

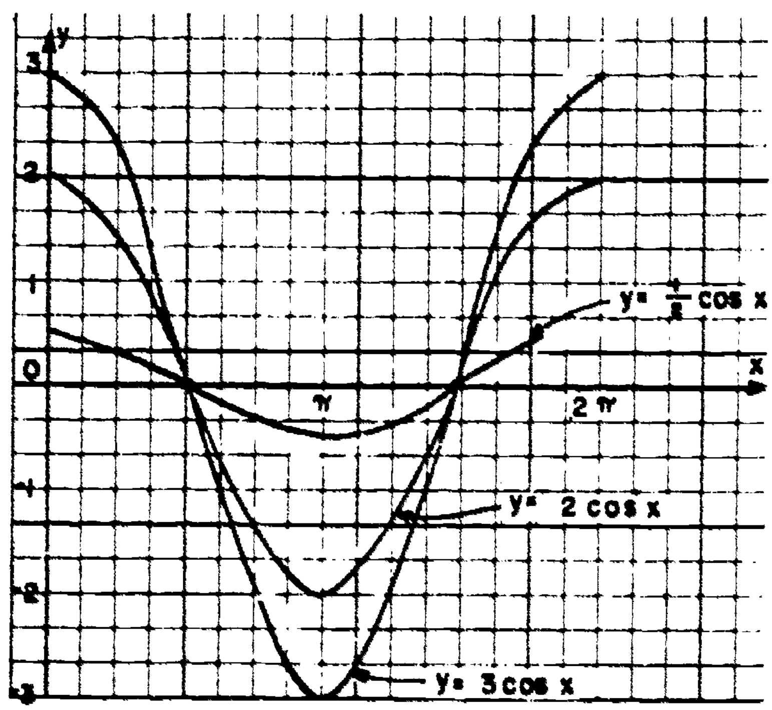
Answers to Exercises 2

1. (a) $f(3\pi) = f(\pi) = -1$
   (b) $f(\frac{7\pi}{3}) = f(\frac{\pi}{3}) = \frac{1}{2}$
   (c) $f(\frac{9\pi}{2}) = f(\frac{\pi}{2}) = 0$
   (d) $f(\frac{25\pi}{6}) = f(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$
   (e) $f(-7\pi) = f(\pi) = -1$
   (f) $f(-\frac{10\pi}{3}) = f(\frac{2\pi}{3}) = -\frac{1}{2}$

2. (a) $f(\pi) = 0$
   (b) $f(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$
   (c) $f(\frac{\pi}{2}) = 1$
   (d) $f(\frac{\pi}{6}) = \frac{1}{2}$
   (e) $f(\pi) = 0$
   (f) $f(\frac{2\pi}{3}) = \frac{\sqrt{3}}{2}$

3. (a) $x = \frac{\pi}{4} + n\pi$
   (b) $x = \frac{3\pi}{4} + n\pi$
   (c) $x = n\pi$
   (d) for all values of x

4.



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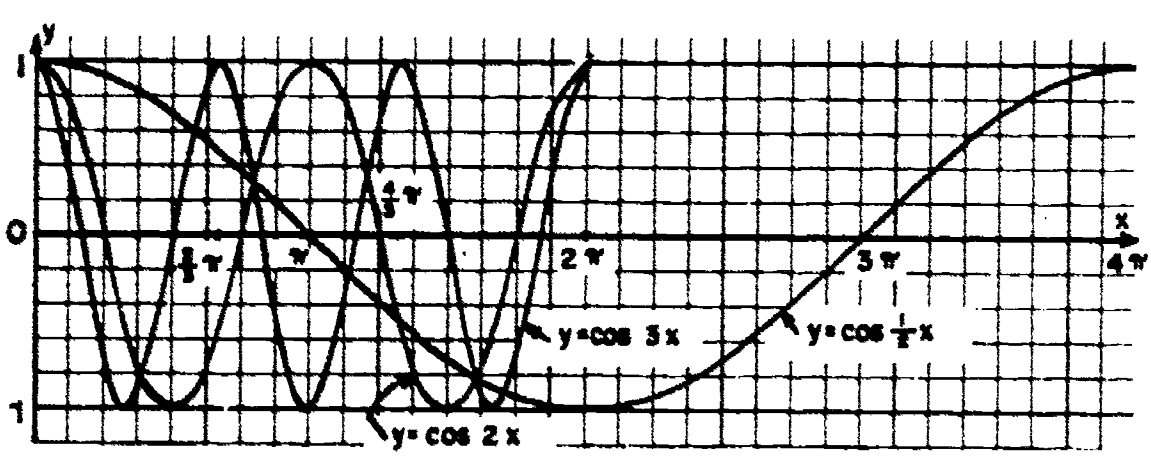

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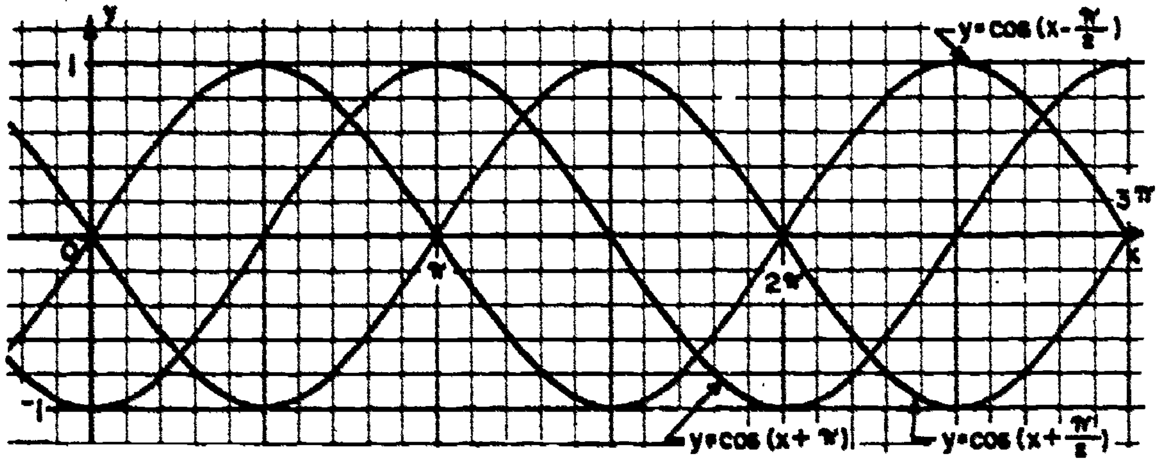


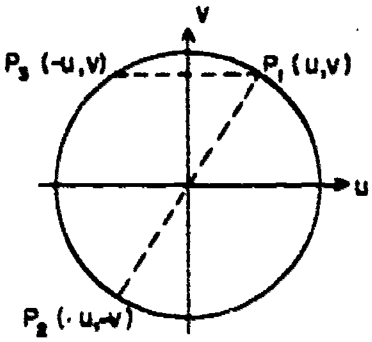
7. (a) The values of the ordinate are multiplied by k.

(b) The period of the graph is $\frac{2\pi}{k}$.

(c) The graph is shifted to the left by the amount $x = k$.

8. $\cos\left(x - \frac{\pi}{2}\right) = \sin x$

9. (a) P_1 and P_2 are symmetric with respect to the origin.



$P_1 = \rho(x) = (u,v)$,

$P_2 = \rho(x - \pi) = \rho(x + \pi)$
$= (-u,-v)$.

Hence, $\cos x = -\cos(x - \pi)$
$= -\cos(x + \pi)$,

and

$\sin x = -\sin(x - \pi)$
$= -\sin(x + \pi)$.

(b) P_1 and P_3 are symmetric with respect to the v-axis.

$P_1 = \rho(x) = (u,v)$,

$P_3 = \rho(-x - \pi) = \rho(-x + \pi) = (-u,v)$.

Hence, $\cos x = -\cos(-x - \pi) = -\cos(-x + \pi)$,

and $\sin x = \sin(-x - \pi) = \sin(-x + \pi)$.

---

3. Angle and Angle Measure.

This is probably review material for most students at this level. Formulas (1) and (2) are the standard radian-degree relationships and the exercises are routine drill in going from one to the other.

Answers to Exercises 3

1. (a) 120° (b) 30° (c) -120° (d) 210° (e) 360° (f) 150° (g) 480° (h) 648° (i) 585°

2. (a) $\frac{3\pi}{2}$ (b) $-\frac{\pi}{6}$ (c) $\frac{3\pi}{4}$ (d) $\frac{8\pi}{3}$ (e) $\frac{13\pi}{12}$ (f) $-\frac{7\pi}{12}$ (g) $\frac{9\pi}{2}$ (h) $\frac{19\pi}{18}$ (i) $\frac{\pi}{10}$

3. $\alpha = \frac{2A}{r^2} = \frac{2 \cdot 9\pi}{9} = 2\pi$

4. $A = \frac{\alpha r^2}{2} = \frac{(\frac{3}{2})\pi \cdot 4}{2} = 3\pi$ square units

5. (a) Since $90^\circ = 100$ "units", $1^\circ = \frac{10}{9}$ "units".

   (b) Since $\frac{\pi}{2} = 100$ "units", 1 radian $= \frac{200}{\pi}$ "units".

   (c) $\alpha = \frac{s}{r} = \frac{2r}{r} = 2$ radians; hence $\alpha = \frac{400}{\pi}$ "units".

---

4. Uniform Circular Motion.

This unit should be taught with care, since the material included will be used in Section 8. In dealing with sin and cos as time functions, we use ωt where ω is the angular velocity, because this is the form in which it appears in most scientific applications. Since up to this point we have dealt with functions connected with an arc length x, you should spend a little time familiarizing the student with ωt.

The device used in the text to visualize the behavior of a wave is only one of several which you may wish to try. Most currently available trigonometry texts have some such approach to the problem, and you should supplement the textual explanation with any other means you feel appropriate.

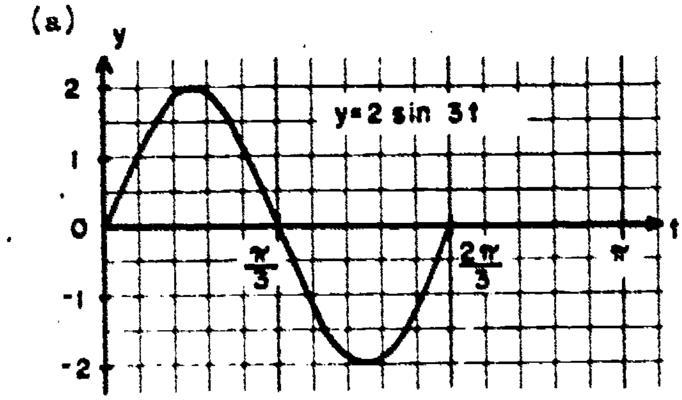
We chose the acoustical example to build upon since the addition of pressures is intuitively simple.

When we use a graph to enhance the student's understanding of a function which maps real numbers into real numbers, we give a true picture of the function only when we use the same scale on both axes. We have followed this practice in most of the graphs of this section of the text. On the other hand, it is sometimes desirable to distort the graph by using different scales in order to show important details which might otherwise be indistinct or confused, and when we graph an equation which describes the relationship between two physical quantities, the question of equal scales may be meaningless. If the pressure p at time t is given by an equation of the form $p = P \cos (\omega t + \alpha)$, we cannot use the same scale on the p-axis as on the t-axis because there is no common measure for time and pressure. Because this situation is one of common occurrence in applications of the circular functions, we have not always insisted on the equal-scales principle. See, for example, Figure 21 and many of the graphs in this section of the commentary.

Answers to Exercises 4

1. See graph. (Note scales.)
   The graph of $p = 3 \cos \pi t + 4 \sin \pi t$ is periodic, with period 2, since corresponding points on the graph are 2, 4, 6, ..., 2n units apart when measured along the t-axis. (Periods of $3 \cos \pi t$ and $4 \sin \pi t$ same as period of p.)

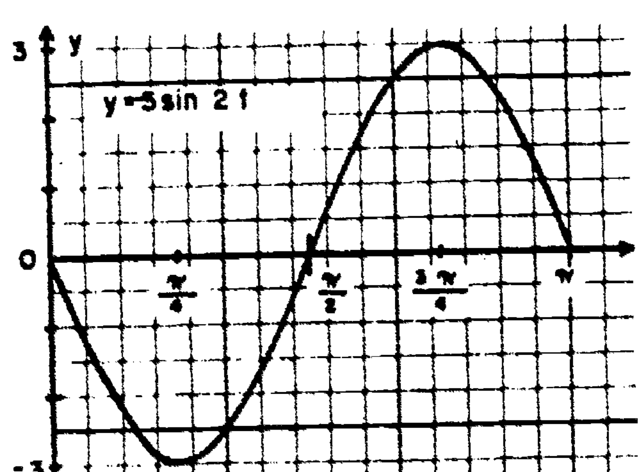
2. (a)



The period is $\frac{2\pi}{3}$.

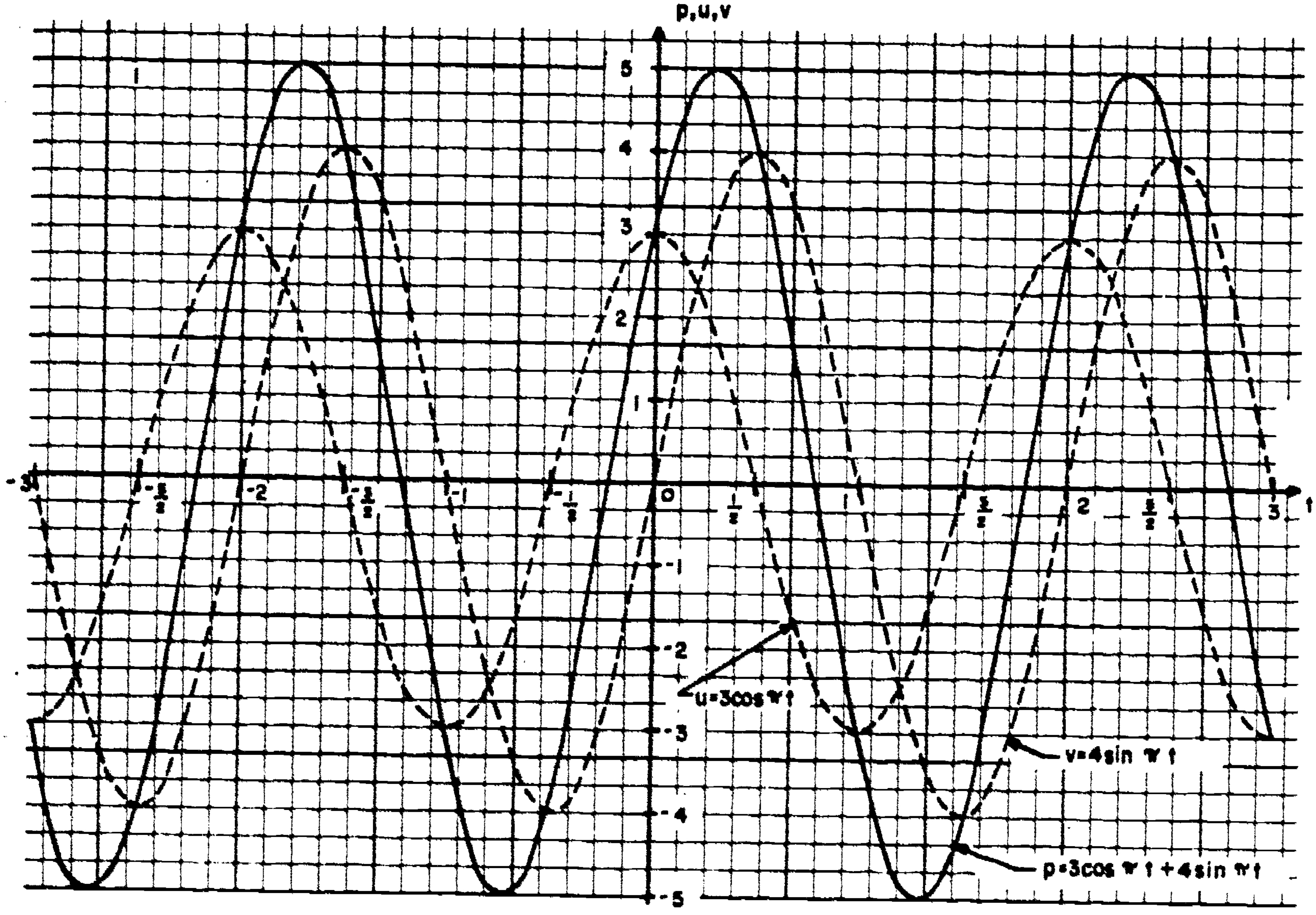
The range is $-2 \leq y \leq 2$.

(b)



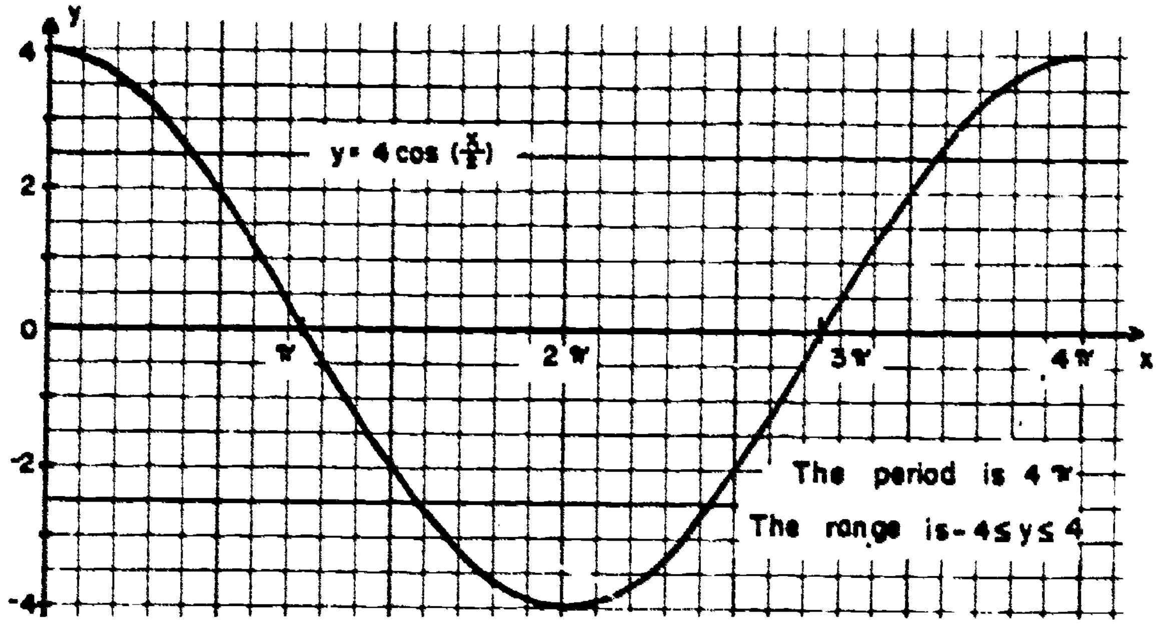
The period is π.

The range is $-3 \leq y \leq 3$.

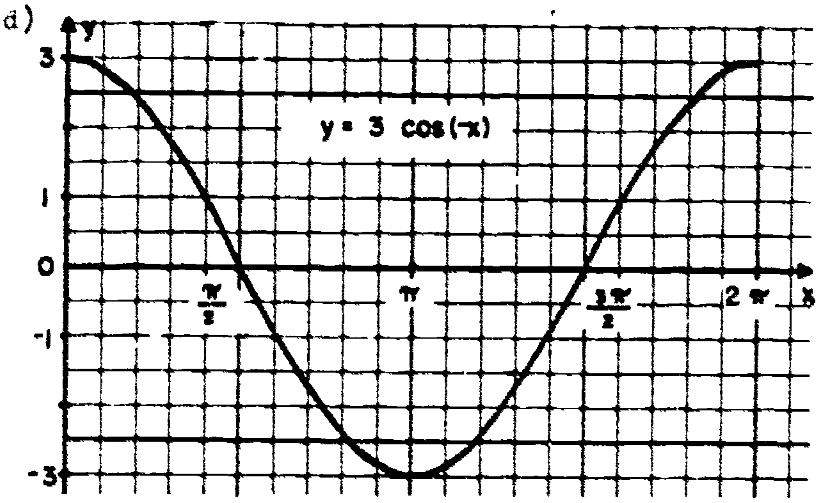


p,u,v
I
u=3cos πt
v=4sin πt
p=3cos πt + 4sin πt
t

(c)



(d)

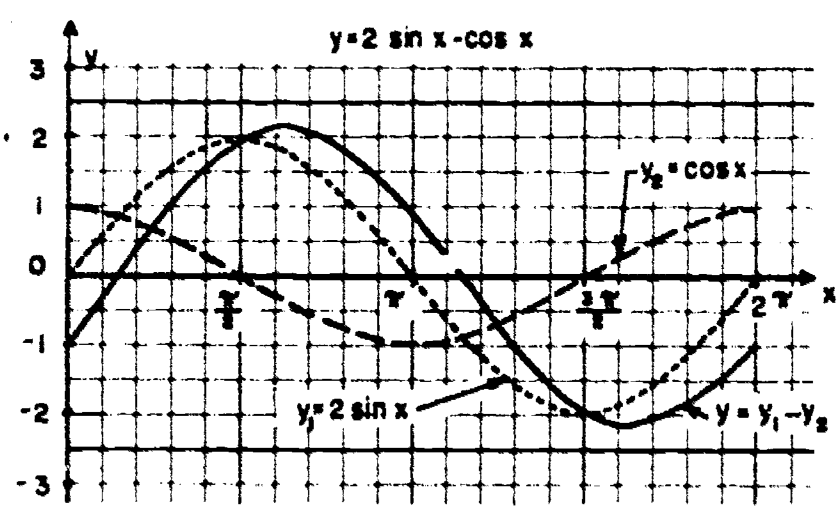


The period is 2π.

The range is

$-3 \le y \le 3$.

(e)



The period is 2π. The range is $-\sqrt{5} \le y \le \sqrt{5}$ (found exactly by the methods of the calculus or those of Section 8 or approximately by the graph).

5. Vectors and Rotations.

We chose the vector approach to the addition formulas for two reasons. First, it should be a means of deriving these relationships different from any which the student has previously encountered. Second, it is an extremely simple and efficient means of obtaining these relationships. We do not, of course, intend this to be a thorough treatment of vectors.

We anticipate that the idea of a rotation as a function, and its effect on a vector, will have to be explained very carefully. You should do a lot of blackboard work here, giving a variety of simple manipulative illustrations. By using chalk of different colors, you can probably improve on some of the figures (such as Figure 26, for example) in the pamphlet. Show vectors rotated in both directions; illustrate rotations followed by rotations; show the rotations of the components of the vector as the vector rotates; in general, make sure that the ideas involved and the symbolism expressing the ideas are clear.

Exercises 5

You may wish to devise additional drill exercises in the use of rotation. Exercises 1, 2, 3, and 4 are cases in point and such problems are easy to make up.

Answers to Exercises 5

1. $\vec{T} = (\frac{\sqrt{2}}{2})\vec{U} + (\frac{\sqrt{2}}{2})\vec{V}, \quad x = \frac{\pi}{4}$

2. (a) $\vec{T} = (-\frac{1}{2})\vec{U} + (-\frac{\sqrt{3}}{2})\vec{V}, \quad x = \frac{4\pi}{3}$

   (b) $\vec{T} = (-\frac{\sqrt{3}}{2})\vec{U} + (-\frac{1}{2})\vec{V}, \quad x = \frac{7\pi}{6}$

3. (a) $f(\vec{U}) = (0)\vec{U} + (-1)\vec{V} = -\vec{V}$

   (b) $f(\vec{U}) = \vec{U}$

4. (a) $f(\vec{U}) = (\frac{\sqrt{2}}{2})\vec{U} + (\frac{\sqrt{2}}{2})\vec{V}$

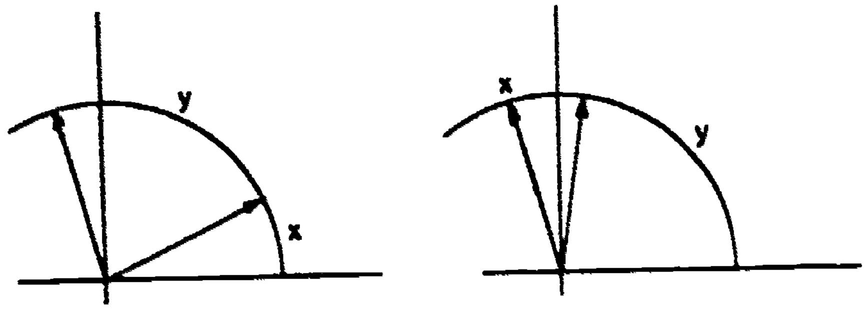
   (b) $f(\vec{U}) = (\frac{1}{2})\vec{U} + (\frac{\sqrt{3}}{2})\vec{V}$

5. $f(\vec{U}) = (-\frac{\sqrt{2}}{2})\vec{U} + (\frac{\sqrt{2}}{2})\vec{V}$

6. $\frac{3\pi}{4} = \frac{\pi}{4} + \frac{\pi}{2}$, and the result follows from (7).

7. Let f correspond to a rotation x, and g, to a rotation y. Then x and y are real numbers; hence, $y + x = x + y$, and the desired result follows.

Geometrically, the result means that a rotation through arc x followed by a rotation through arc y is equivalent to a rotation through arc y followed by a rotation through arc x.



8. Since $\vec{V} = g(\vec{U})$,

$$f(\vec{V}) = f\big(g(\vec{U})\big) = (fg)(\vec{U})$$
$$= (gf)(\vec{U}), \text{ by the result of Exercise 7.}$$

9. From Exercise 8, $f(\vec{V}) = f\big(g(\vec{U})\big)$

$$= g\big(f(\vec{U})\big)$$
$$= g(u\vec{U} + v\vec{V})$$
$$= ug(\vec{U}) + vg(\vec{V})$$
$$= u\vec{V} - v\vec{U}$$

since $g(\vec{U}) = \vec{V}$ and $g(\vec{V}) = -\vec{U}$.

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6. Addition Formulas for Sine and Cosine.

The derivation of $\cos (x + y)$ and $\sin (x + y)$ is usually accomplished either by geometric considerations in the first quadrant (which then involve a great deal of work to generalize), or by use of the distance formula. As remarked before, we feel the vector approach to be new and instructive and, in essence, simpler than either of the aforementioned. We include the page on the relation to complex numbers to show still another means of deriving these formulas.

Exercises 6

The exercises are, in general, identities; applications of the sum and difference formulas. You may wish to illustrate a few samples on the blackboard before asking the students to work the exercises. Exercises 4, 5, and 6 are important since the tangent function appears here for the first time and some of its properties are investigated. You should be sure to cover these exercises at some point in the work.

Answers to Exercises 6

1. (a) $\cos(\frac{\pi}{2} - x) = \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x$

$= 0 + \sin x$

$= \sin x$

(b) $\sin(\frac{\pi}{2} - x) = \sin\frac{\pi}{2}\cos x - \cos\frac{\pi}{2}\sin x$

$= \cos x - 0$

$= \cos x$

(c) $\cos(x + \frac{\pi}{2}) = \cos x\cos\frac{\pi}{2} - \sin x\sin\frac{\pi}{2}$

$= 0 - \sin x$

$= -\sin x$

(d) $\sin(x + \frac{\pi}{2}) = \sin x\cos\frac{\pi}{2} + \cos x\sin\frac{\pi}{2}$

$= 0 + \cos x$

$= \cos x$

(e) $\cos(\pi - x) = \cos\pi\cos x + \sin\pi\sin x$

$= (-1)\cos x + 0$

$= -\cos x$

(f) $\sin(\pi - x) = \sin\pi\cos x - \cos\pi\sin x$

$= 0 - (-1)\sin x$

$= \sin x$

(g) $\cos(\frac{3\pi}{2} + x) = \cos\frac{3\pi}{2}\cos x - \sin\frac{3\pi}{2}\sin x$

$= 0 - (-1)\sin x$

$= \sin x$

(h) $\sin(\frac{3\pi}{2} + x) = \sin\frac{3\pi}{2}\cos x + \cos\frac{3\pi}{2}\sin x$

$= (-1)\cos x + 0$

$= -\cos x$

(i) $\sin(\frac{\pi}{4} + x) = \sin\frac{\pi}{4}\cos x + \cos\frac{\pi}{4}\sin x$

$= (\frac{\sqrt{2}}{2})(\cos x + \sin x);$

$\cos(\frac{\pi}{4} - x) = \cos\frac{\pi}{4}\cos x + \sin\frac{\pi}{4}\sin x$

$= (\frac{\sqrt{2}}{2})(\cos x + \sin x).$

Hence, $\sin(\frac{\pi}{4} + x) = \cos(\frac{\pi}{4} - x)$.

2. $\sin(x - y) = \sin\big(x + (-y)\big)$
$= \sin x \cos(-y) + \cos x \sin(-y)$
$= \sin x \cos y - \cos x \sin y$

*3. Formula 10: $\cos(x - y) = \cos x \cos y + \sin x \sin y$

To derive 7: $\cos(x + y) = \cos\big(x - (-y)\big)$
$= \cos x \cos(-y) + \sin x \sin(-y)$
$= \cos x \cos y - \sin x \sin y$

To derive 8: $\sin(x + y) = \cos\left(\frac{\pi}{2} - (x + y)\right)$ from Exercise 1(a).
$= \cos\left(\left(\frac{\pi}{2} - x\right) - y\right)$
$= \cos\left(\frac{\pi}{2} - x\right) \cos y + \sin\left(\frac{\pi}{2} - x\right) \sin y$

To simplify $\cos\left(\frac{\pi}{2} - x\right)$ and $\sin\left(\frac{\pi}{2} - x\right)$, use Exercise 1(a).

$\cos\left(\frac{\pi}{2} - x\right) = \sin x$

$\sin\left(\frac{\pi}{2} - x\right) = \cos\left(\frac{\pi}{2} - \left(\frac{\pi}{2} - x\right)\right)$
$= \cos x.$

Hence, $\cos\left(\frac{\pi}{2} - x\right) \cos y + \sin\left(\frac{\pi}{2} - x\right) \sin y$

becomes $\sin x \cos y + \cos x \sin y.$

Therefore, $\sin(x + y) = \sin x \cos y + \cos x \sin y.$

To derive 11, use 8 just obtained:

$\sin(x - y) = \sin\big(x + (-y)\big)$
$= \sin x \cos(-y) + \cos x \sin(-y)$
$= \sin x \cos y - \cos x \sin y.$

4. tan: $x \to \frac{\sin x}{\cos x}$ $\left(x \neq \pm\frac{\pi}{2} + 2n\pi\right)$

To prove that tan is periodic with period π, we must prove that $\tan(x + \pi) = \tan x.$

From the definition, $\tan(x + \pi) = \frac{\sin(x + \pi)}{\cos(x + \pi)}$
$= \frac{-\sin x}{-\cos x}$ (from Exercises 2, 9(a))
$= \tan x.$

Now, $\tan\left(\pm\frac{\pi}{2} + 2n\pi\right) = \dfrac{\sin\left(\pm\frac{\pi}{2} + 2n\pi\right)}{\cos\left(\pm\frac{\pi}{2} + 2n\pi\right)}.$

But the denominator of this fraction is zero and therefore the values of $\tan\left(\pm\frac{\pi}{2} + 2n\pi\right)$ are undefined.

4

5. $\tan(x \pm y) = \frac{\sin(x \pm y)}{\cos(x \pm y)} = \frac{\sin x \cos y \pm \cos x \sin y}{\cos x \cos y \mp \sin x \sin y}$

Dividing numerator and denominator by $\cos x \cos y$,

$$= \frac{\frac{\sin x \cos y}{\cos x \cos y} \pm \frac{\cos x \sin y}{\cos x \cos y}}{\frac{\cos x \cos y}{\cos x \cos y} \mp \frac{\sin x \sin y}{\cos x \cos y}}$$

$$= \frac{\frac{\sin x}{\cos x} \pm \frac{\sin y}{\cos y}}{1 \mp \frac{\sin x}{\cos x} \cdot \frac{\sin y}{\cos y}}$$

$$= \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}.$$

6. $\tan(\pi - x) = \frac{\tan \pi - \tan x}{1 + \tan \pi \tan x} = \frac{0 - \tan x}{1 + 0} = -\tan x$

$\tan(\pi + x) = \frac{\tan \pi + \tan x}{1 - \tan \pi \tan x} = \frac{0 + \tan x}{1 - 0} = \tan x$

$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin x}{\cos x} = -\tan x$

7. $\sin 2x = \sin(x + x) = \sin x \cos x + \cos x \sin x = 2 \sin x \cos x$

$\cos 2x = \cos(x + x) = \cos x \cos x - \sin x \sin x = \cos^2 x - \sin^2 x$

$\tan 2x = \tan(x + x) = \frac{\tan x + \tan x}{1 - \tan x \tan x} = \frac{2 \tan x}{1 - \tan^2 x}$

8. $\sin 3x = \sin(2x + x) = \sin 2x \cos x + \cos 2x \sin x$

$= 2 \sin x \cos^2 x + (\cos^2 x - \sin^2 x)\sin x$

$= 3 \sin x \cos^2 x - \sin^3 x$

9. $\cos 2x = 1 - 2 \sin^2 x$

Let $x = \frac{y}{2}$.

$\cos y = 1 - 2 \sin^2 \frac{y}{2}$

$\sin^2 \frac{y}{2} = \frac{1 - \cos y}{2}$

$\sin \frac{y}{2} = \pm\sqrt{\frac{1 - \cos y}{2}}$

10. $\cos 2x = 2 \cos^2 x - 1$

Let $x = \frac{y}{2}$.

$\cos y = 2 \cos^2 \frac{y}{2} - 1$

$\cos^2 \frac{y}{2} = \frac{1 + \cos y}{2}$

$\cos \frac{y}{2} = \pm\sqrt{\frac{1 + \cos y}{2}}$

11. $\tan \frac{y}{2} = \frac{\sin \frac{y}{2}}{\cos \frac{y}{2}} = \pm\sqrt{\frac{1 - \cos y}{1 + \cos y}}$

Multiply the right member by $\sqrt{\frac{1 - \cos y}{1 - \cos y}}$;

$$\tan \frac{y}{2} = \pm\sqrt{\frac{(1 - \cos y)^2}{1 - \cos^2 y}}$$

$$= \pm\sqrt{\frac{(1 - \cos y)^2}{\sin^2 y}}$$

$$= \frac{1 - \cos y}{\sin y}.$$

Note: The result given is correct since $\tan \frac{y}{2}$ and $\frac{1 - \cos y}{\sin y}$ agree in sign for all possible combinations of sign of $\cos y$ and $\sin y$.

Multiply the right member by $\sqrt{\frac{1 + \cos y}{1 + \cos y}}$;

$$\tan \frac{y}{2} = \pm\sqrt{\frac{1 - \cos^2 y}{(1 + \cos y)^2}} = \pm\sqrt{\frac{\sin^2 y}{(1 + \cos y)^2}} = \frac{\sin y}{1 + \cos y}.$$

(See previous note.)

Alternatively,

$$\tan \frac{y}{2} = \frac{\sin \frac{y}{2}}{\cos \frac{y}{2}} = \frac{2 \sin \frac{y}{2} \cos \frac{y}{2}}{2 \cos^2 \frac{y}{2}} = \frac{\sin y}{1 + \cos y}.$$

---

7. Construction and Use of Tables of Circular Functions.

Since this material is largely in the nature of a review, you will probably not wish to spend much time on it. The table of decimal fractions of $\frac{\pi}{2}$ will be new to the student, but we use it as we do any other table, and it should cause no difficulty.

Answers to Exercises 7a

1. Table I is not folded because the values of x are given in such a way that they are not symmetrical about $x = \frac{\pi}{4} \approx 0.785$. For example,

$$\cos 0.60 = \sin (\tfrac{\pi}{2} - 0.60).$$

Since we are using radian measure, $\frac{\pi}{2}$ is irrational, and hence we would have to use an irrational interval (as is done in Table II) to get a symmetric table.

$$\cos 0.60 \approx \sin (1.57 - 0.60)$$
$$\approx \sin 0.97.$$

From the table, $\cos 0.60 = 0.8253$ and $\sin 0.97 = 0.8249$. The values of $\cos 0.60$ and $\sin 0.97$ would have to be the same if the table could be folded.

2. (a) $\sin 0.73 \approx 0.6669$, $\cos 0.73 \approx 0.7452$

   (b) $\sin(-5.17) = \sin(-5.17 + 2\pi) \approx \sin 1.11 \approx 0.8957$

   $\cos(-5.17) \approx \cos 1.11 \approx 0.4447$

   (c) $\sin 1.55 \approx 0.9998$, $\cos 1.55 \approx 0.0208$

   (d) $\sin 6.97 = \sin(6.97 - 2\pi) \approx \sin 0.69 \approx 0.6365$

   $\cos 6.97 \approx \cos 0.69 \approx 0.7712$

3. (a) $\sin x \approx 0.1099$, $x \approx 0.11$

   (b) $\cos x \approx 0.9131$, $x \approx 0.42$

   (c) $\sin x \approx 0.6495$, $x \approx 0.71$

   (d) $\cos x \approx 0.5403$, $x \approx 1.00$

Note: Hereafter we use "$=$" for "$\approx$".

4. (a) $\sin 0.31(\frac{\pi}{2}) = 0.468$, $\cos 0.31(\frac{\pi}{2}) = 0.884$

   (b) $\sin 0.79(\frac{\pi}{2}) = 0.946$, $\cos 0.79(\frac{\pi}{2}) = 0.324$

   (c) $\sin 0.62(\frac{\pi}{2}) = 0.827$, $\cos 0.62(\frac{\pi}{2}) = 0.562$

   (d) $\sin 0.71(\frac{\pi}{2}) = 0.898$, $\cos 0.71(\frac{\pi}{2}) = 0.440$

5. (a) $\sin \omega t = 0.827$, $t = 0.62$

   (b) $\cos \omega t = 0.905$, $t = 0.28$

   (c) $\sin \omega t = 0.475$, $t = 0.315$

   (d) $\cos \omega t = 0.795$, $t = 0.415$

6. (a) $\sin 45^\circ = 0.707$, $\cos 45^\circ = 0.707$

   (b) $\sin 73^\circ = 0.956$, $\cos 73^\circ = 0.292$

   (c) $\sin 36.2^\circ = 0.591$, $\cos 36.2^\circ = 0.807$

   (d) $\sin 81.5^\circ = 0.989$, $\cos 81.5^\circ = 0.148$

7. (a) $\sin x = 0.629$, $x = 39^\circ$

   (b) $\cos x = 0.991$, $x = 7.7^\circ$

   (c) $\sin x = 0.621$, $x = 38.4^\circ$

   (d) $\cos x = 0.895$, $x = 26.5^\circ$

Answers to Exercises 7b

1. $\sin 1.73 = \sin(\pi - 1.73) = \sin 1.41 = 0.9871$ (Table I)

2. $\cos 1.3\pi = -\cos(1.3\pi - \pi) = -\cos 0.3\pi = -\cos 0.60(\frac{\pi}{2}) = -0.588$ (Table II)

3. $\sin(-.37) = -\sin .37 = -0.3616$ (Table I)

4. $\sin(-.37\pi) = -\sin .74(\frac{\pi}{2}) = -0.918$ (Table II)

5. $\cos 2.8\pi = \cos(2.8\pi - 2\pi) = \cos 0.8\pi = -\cos(\pi - 0.8\pi) = -\cos 0.2\pi$
$= -\cos 0.4(\frac{\pi}{2}) = -0.809$ (Table II)

6. $\cos 1.8\pi = \cos(2\pi - 1.8\pi) = \cos 0.2\pi = 0.809$ (from Exercise 5)

7. $\cos 3.71 = -\cos(3.71 - \pi) = -\cos 0.57 = -0.8419$ (Table I)

8. $\sin 135^\circ = \sin(180^\circ - 135^\circ) = \sin 45^\circ = 0.707$ (Table III)

9. $\cos(-135^\circ) = -\cos(180^\circ - 135^\circ) = -\cos 45^\circ = -0.707$ (Table III)

10. $\sin 327^\circ = -\sin(360^\circ - 327^\circ) = -\sin 33^\circ = -0.545$ (Table III)

11. $\cos(-327^\circ) = \cos(360^\circ - 327^\circ) = \cos 33^\circ = 0.839$ (Table III)

12. $\cos 12.4\pi = \cos(12.4\pi - 12\pi) = \cos 0.4\pi = \cos 0.8(\frac{\pi}{2}) = 0.309$ (Table II)

13. $\sin 12.4 = -\sin(4\pi - 12.4) = -\sin 0.16 = -0.1593$ (Table I)

*14. $\cos(\sin .3\pi) = \cos\left(\sin 0.6(\frac{\pi}{2})\right) = \cos 0.809$ (Table II)
$= 0.6902$ (Table I)

*15. $\sin(\sin .7) = \sin 0.6442 = 0.6005$ (Table I)

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8. Pure Waves: Frequency, Amplitude, and Phase.

We chose cos as our standard wave because its first peak occurs at 0. Since we are using peaks to discuss phase, cos serves better than sin which peaks first at $\frac{\pi}{2}$. The phase α is often called a lag ($\alpha > 0$) or a lead ($\alpha < 0$); if $\alpha = 0$, the wave is in phase with the standard. By using $0 \leq \alpha < 2\pi$, we avoid all mention of a wave "leading". This is a departure from the conventional, in that most sciences which have occasion to discuss lead or lag use both. You may wish to explore this idea by examining with the class the effect of using $-\pi \leq \alpha < \pi$, and show that $\alpha < 0$ represents a lead in the sense that $\alpha > 0$ represents a lag.

Note that α is not, in general, the abscissa of the first maximum. In fact, $A \cos(\omega t - \alpha)$ $(A > 0)$ reaches a peak when $\omega t - \alpha = 0$ or $t = \frac{\alpha}{\omega}$; hence, the abscissa of the first maximum is $\frac{\alpha}{\omega}$.

Answers to Exercises 8

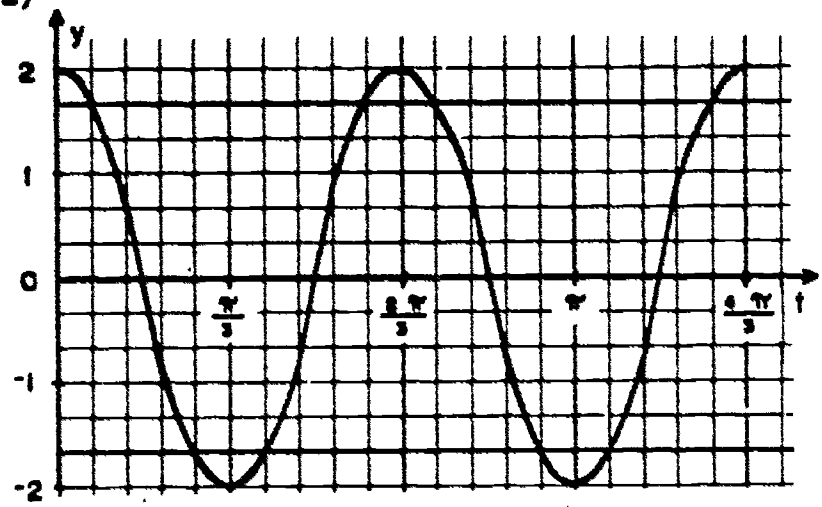
1. Since $p \approx 5 \cos(\pi t - 0.927)$,

$p = 0$ if $\pi t - 0.927 \approx \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

Hence, $t \approx \frac{0.927}{\pi} + \frac{1}{2}$ or $\frac{0.927}{\pi} + \frac{3}{2}$.

Since $\frac{0.927}{\pi} \approx 0.29$, $t \approx 0.79$ or 1.79.

2. (a)

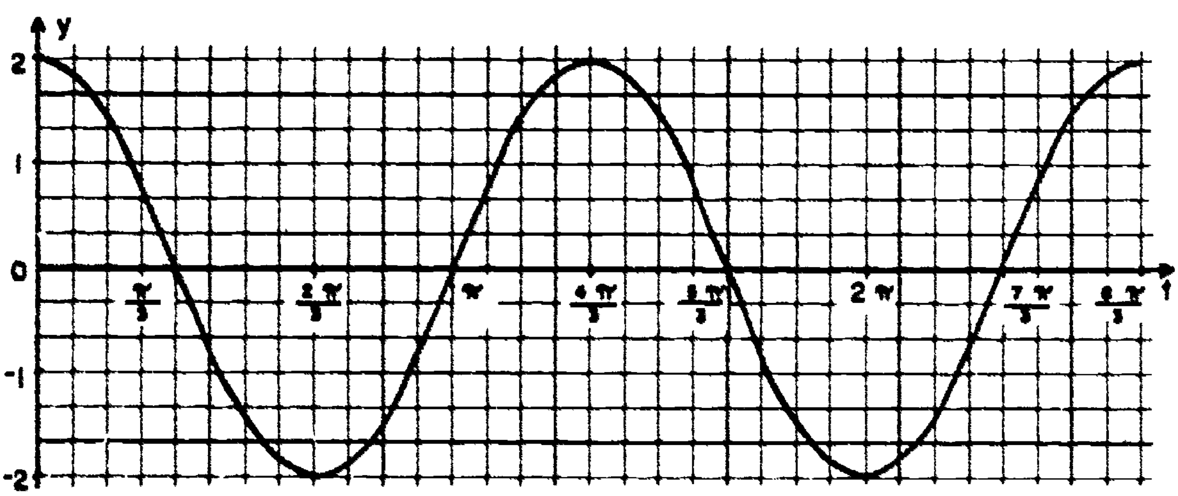


Amplitude = 2,

Period = $\frac{2\pi}{3}$,

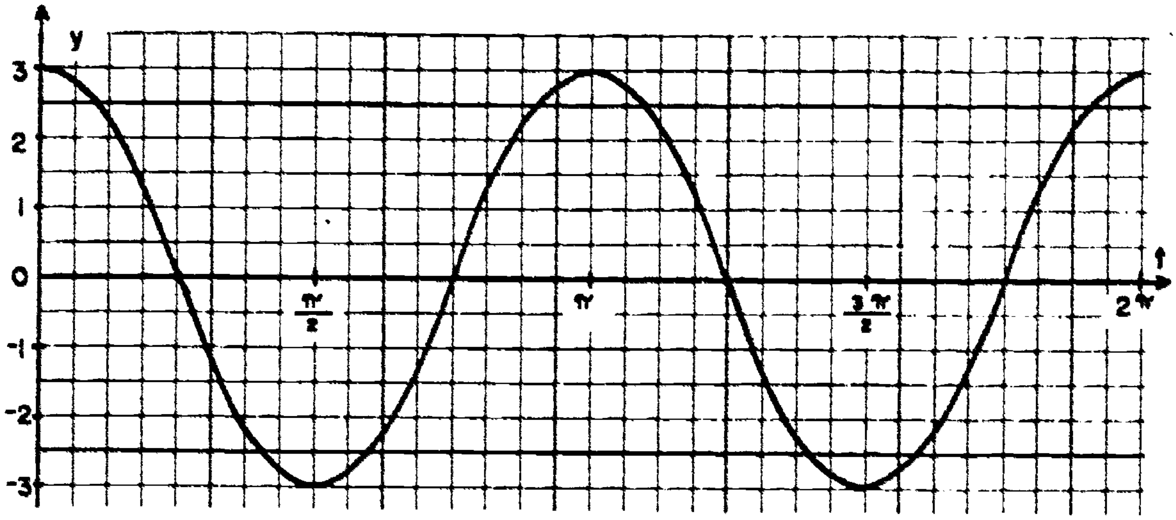
Phase = 0.

(b)



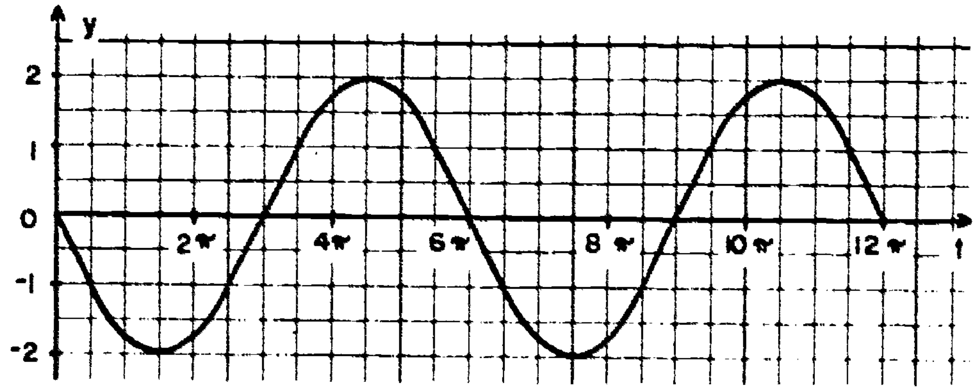
Amplitude = 2, Period = $\frac{4\pi}{3}$, Phase = 0.

(c)



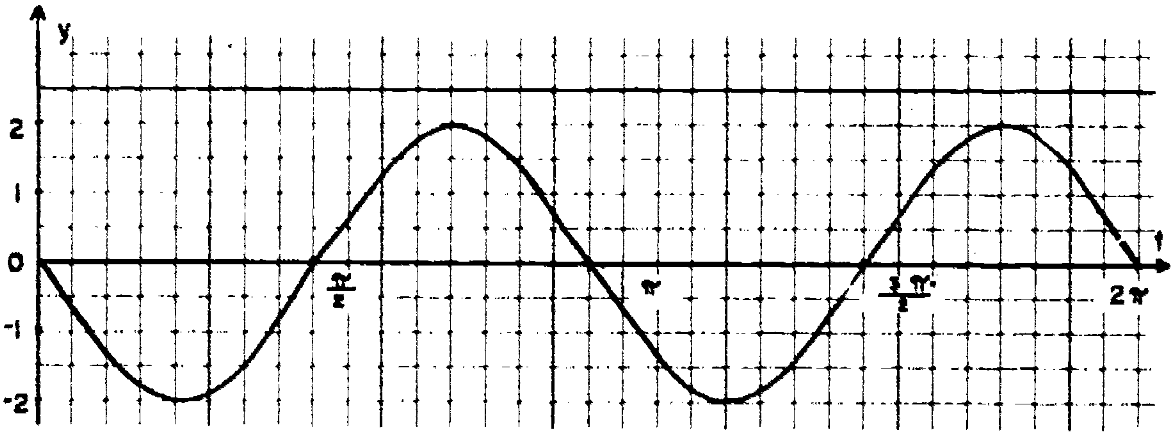
Amplitude = 3, Period = π, Phase = 0.

(d)



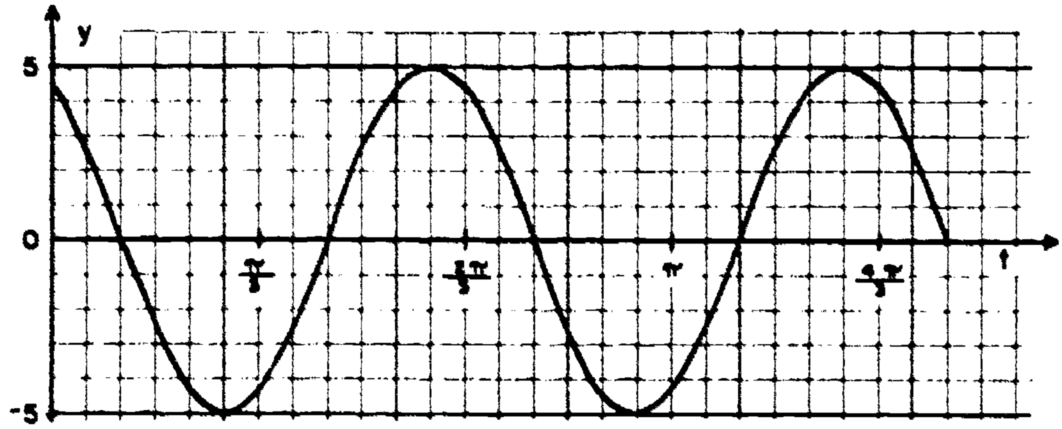
Amplitude = 2, Period = 6π, Phase = $\frac{1}{3} \cdot \frac{9\pi}{2} = \frac{3\pi}{2}$.

(e)



Amplitude = 2, Period = π, Phase = $2 \cdot \frac{3\pi}{4} = \frac{3\pi}{2}$.

(f)



Amplitude = 5, Period = $\frac{2\pi}{3}$, Phase = $3 \cdot \frac{11\pi}{18} = \frac{11\pi}{6}$.

3. (a) $y = A\cos(\omega t - \alpha) = A\cos\omega t\cos\alpha + A\sin\omega t\sin\alpha$

$y = 4\sin\pi t - 3\cos\pi t$

$A\sin\alpha = 4, \quad A\cos\alpha = -3$

$A^2(\sin^2\alpha + \cos^2\alpha) = 16 + 9$

$A^2 = 25, \quad A = 5$

$\sin\alpha = \frac{4}{5}, \quad \cos\alpha = -\frac{3}{5}$

$\alpha \approx \pi - 0.927 \approx 2.215$

Answer: $y = 5\cos(\pi t - 2.215)$

(b) $y = -4\sin\pi t + 3\cos\pi t$

$A\sin\alpha = -4, \quad A\cos\alpha = 3, \quad A = 5.$

$\sin\alpha = -\frac{4}{5}, \quad \cos\alpha = \frac{3}{5}$

$\alpha \approx 2\pi - 0.927 \approx 5.357$

Answer: $y = 5\cos(\pi t - 5.357)$

(c) $y = -4\sin\pi t - 3\cos\pi t$

$A = 5, \quad \sin\alpha = -\frac{4}{5}, \quad \cos\alpha = -\frac{3}{5}$

$\alpha \approx \pi + 0.927 \approx 4.069$

Answer: $y = 5\cos(\pi t - 4.069)$

(d) $y = 3\sin\pi t + 4\cos\pi t$

$A = 5, \quad \sin\alpha = \frac{3}{5}, \quad \cos\alpha = \frac{4}{5}$

$\alpha \approx 0.644$

Answer: $y = 5\cos(\pi t - 0.644)$

(e) $y = 3 \sin \pi t - 4 \cos \pi t$

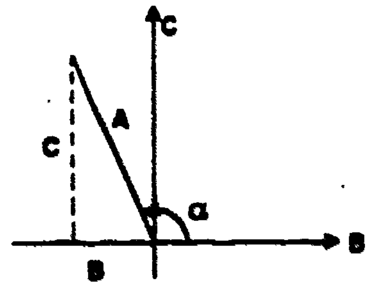
$A = 5, \quad \sin \alpha = \frac{3}{5}, \quad \cos \alpha = -\frac{4}{5}$

$\alpha \approx \pi - 0.644 \approx 2.498$

Answer: $y = 5 \cos (\pi t - 2.498)$

4. $A^2 = B^2 + C^2$

$\sin \alpha = \frac{C}{A}, \quad \cos \alpha = \frac{B}{A}$



Although the directions in this problem do not ask for the values of t at which the maxima and minima occur, they have been included in these solutions in case the question arises.

(a) $A = 5, \quad \sin \alpha = \frac{3}{5}, \quad \cos \alpha = \frac{4}{5}, \quad \alpha \approx 0.644.$

Hence, $3 \sin 2t + 4 \cos 2t \approx 5 \cos (2t - 0.644).$

Maximum value, 5, occurs when $\cos (2t - 0.644) = 1$, or

$$2t - 0.644 = 0, \quad t = 0.322.$$

Minimum value, -5, occurs when $\cos (2t - 0.644) = -1$, or

$$2t - 0.644 = \pi, \quad t \approx 1.893.$$

The period $= \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi.$

Hence, maximum values occur at $t \approx 0.322 + n\pi$ and minimum values at $t \approx 1.893 + n\pi$.

(b) $A = \sqrt{4 + 9} = \sqrt{13}, \quad \sin \alpha = \frac{2}{\sqrt{13}}, \quad \cos \alpha = \frac{-3}{\sqrt{13}},$

$\alpha \approx \pi - 0.589 \approx 2.553.$

Hence, $2 \sin 3t - 3 \cos 3t \approx \sqrt{13} \cos (3t - 2.553).$

The period $= \frac{2\pi}{3}$. Maximum values, $\sqrt{13}$, occur when

$$3t - 2.553 = 0 + 2n\pi, \quad t \approx 0.851 + \frac{2n\pi}{3}.$$

Minimum values, $-\sqrt{13}$, occur when

$$3t - 2.553 = \pi + 2n\pi, \quad t \approx 1.898 + \frac{2n\pi}{3}.$$

(c) $A = \sqrt{1 + 1} = \sqrt{2}, \quad \sin \alpha = -\frac{1}{\sqrt{2}}, \quad \cos \alpha = \frac{1}{\sqrt{2}}, \quad \alpha = \frac{7\pi}{4}.$

Hence, $-\sin (\frac{t}{2}) + \cos (\frac{t}{2}) = \sqrt{2} \cos (\frac{t}{2} - \frac{7\pi}{4}).$

The period $= \frac{2\pi}{\frac{1}{2}} = 4\pi$. Maximum values, $\sqrt{2}$, occur when

$$\frac{t}{2} - \frac{7\pi}{4} = 0 + 2n\pi, \quad t = \frac{7\pi}{2} + 4n\pi.$$

Minimum values, $-\sqrt{2}$, occur when

$$\frac{t}{2} - \frac{7\pi}{4} = \pi + 2n\pi, \quad t = \frac{3\pi}{2} + 4n\pi.$$

5. $A \cos(\omega t - \alpha) + B \cos(\omega t - \beta)$

$= A \cos \omega t \cos \alpha + A \sin \omega t \sin \alpha + B \cos \omega t \cos \beta + B \sin \omega t \sin \beta$

$= (A \cos \alpha + B \cos \beta)\cos \omega t + (A \sin \alpha + B \sin \beta)\sin \omega t$

$= C \cos(\omega t - \gamma)$ when

$$C = \sqrt{(A \sin \alpha + B \sin \beta)^2 + (A \cos \alpha + B \cos \beta)^2},$$

$$\sin \gamma = \frac{A \sin \alpha + B \sin \beta}{C}, \text{ and } \cos \gamma = \frac{A \cos \alpha + B \cos \beta}{C}.$$

Since A, B, α, and β are real numbers, it follows that C is a real number, and it is easy to show that

$$0 \le \sin^2\gamma \le 1, \quad 0 \le \cos^2\gamma \le 1, \quad \sin^2\gamma + \cos^2\gamma = 1,$$

and therefore γ is a real number.

6. (a) From the solution in the text, $t = \frac{0.927}{\pi} \pm \frac{1}{3} + 2n$.

So $t \approx 0.295 \pm 0.333 + 2n$.

The smallest positive value of t is

$$t \approx 0.295 + 0.333 = 0.628.$$

(b) $3 \cos \pi t + 4 \sin \pi t = 5$

$5 \cos(\pi t - 0.927) = 5$

$\cos(\pi t - 0.927) = 1$

This is satisfied when the argument of the cosine is $0 + 2n\pi$.

Therefore, $\pi t - 0.927 = 2n\pi$,

or $$t \approx \frac{0.927}{\pi} + 2n \approx 0.295 + 2n.$$

The smallest positive value of t is

$$t \approx 0.295.$$

(c) $\sin 2t - \cos 2t = 1$

$\sqrt{2} \cos\left(2t - \frac{3\pi}{4}\right) = 1$

$\cos\left(2t - \frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$

This is satisfied when the argument of the cosine is $\pm\frac{\pi}{4} + 2n\pi$.

Therefore, $2t - \frac{3\pi}{4} = \pm\frac{\pi}{4} + 2n\pi$,

or $$t = \frac{3\pi}{8} \pm \frac{\pi}{8} + n\pi.$$

The smallest positive value of t is

$$t = \frac{\pi}{4}.$$

(d) $4 \cos \pi t - 3 \sin \pi t = 0$

$5 \cos (\pi t - 5.640) = 0$

$\cos (\pi t - 5.640) = 0$

This is satisfied when the argument of the cosine is $\pm\frac{\pi}{2} + 2n\pi$.

Therefore, $\pi t - 5.640 = \pm\frac{\pi}{2} + 2n\pi$,

or $$t \approx \frac{5.640}{\pi} \pm \frac{1}{2} + 2n \approx 1.795 \pm 0.5 + 2n.$$

The smallest positive value of t is

$$t \approx 1.795 + 0.5 - 2 \approx 0.295.$$

(e) $4 \cos \pi t + 3 \sen \pi t = 1$

$5 \cos (\pi t - 0.644) = 1$

$\cos (\pi t - 0.644) = 0.2$

This is satisfied when the argument of the cosine is approximately $\pm 1.369 + 2n\pi$ (from Table I).

Therefore, $\pi t - 0.644 \approx \pm 1.369 + 2n\pi$,

or $$t \approx \frac{0.644 \pm 1.369}{\pi} + 2n.$$

The smallest positive value of t is

$$t \approx \frac{0.644 + 1.369}{\pi} + 0 \approx 0.641.$$

7. Given $y = B \cos (\mu t - \beta)$.

We may clearly assume that $0 < \beta < 2\pi$.

1. If μ and B are positive, we set $\mu = \omega$, $B = A$, $\beta = \alpha$.

2. If μ is positive and B is negative, set $\mu = \omega$, $B = -A$.
   Then $y = A\left(-\cos (\omega t - \beta)\right) = A \cos (\omega t - \beta \pm \pi)$.
   If $0 \leq \beta < \pi$, take $\alpha = \beta + \pi$.
   If $\pi \leq \beta < 2\pi$, take $\alpha = \beta - \pi$.

3. If μ is negative, set $\mu = -\omega$.
   Then $y = B \cos (-\omega t - \beta) = B \cos (\omega t + \beta)$
   $= B \cos \left(\omega t - (2\pi - \beta)\right)$
   $= B \cos (\omega t - \beta')$.

Proceed as in 1 and 2.

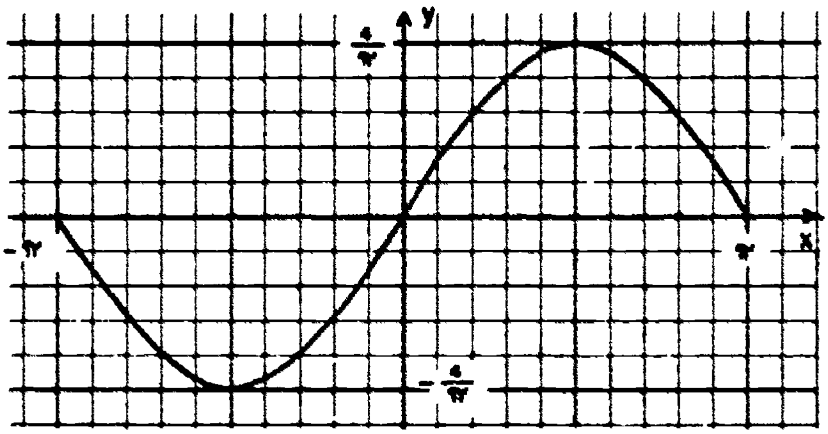
5

9. Analysis of General Waves.

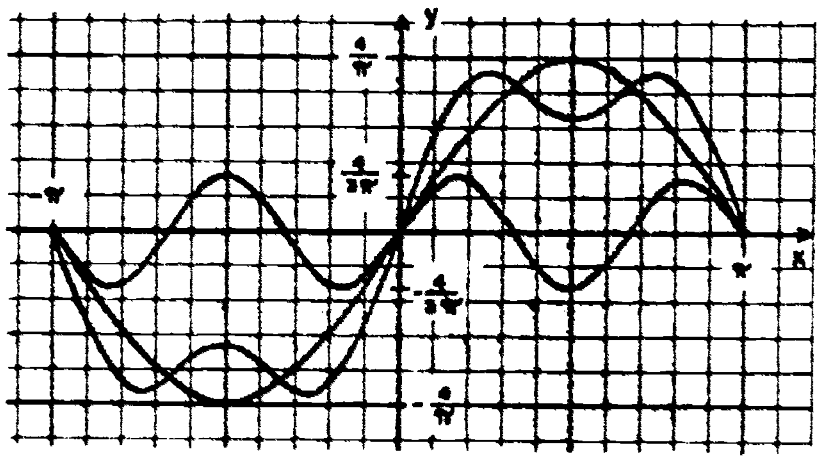
The purpose of this section is to provide the student with some idea of the power of simple circular functions, and to show how they can be used to approximate much more complex periodic functions. We do not intend that the student use Fourier's Theorem, but only that he understand what it says, and what it implies.

Answers to Exercises 9

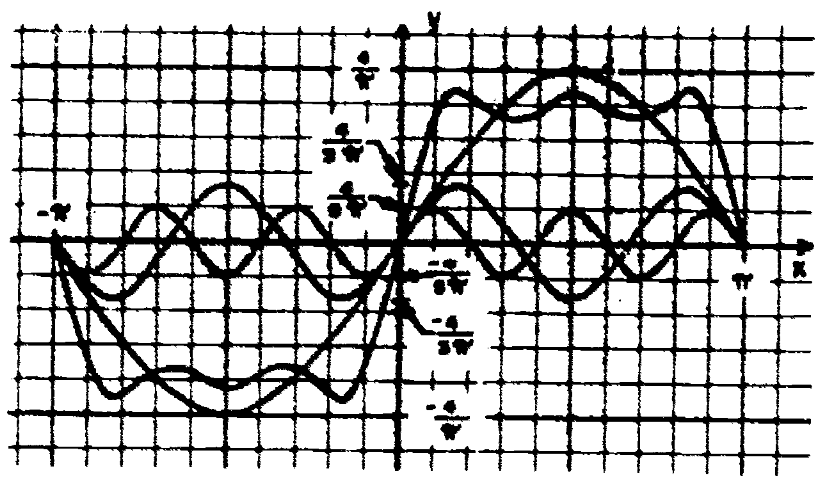
1. (a)



(b)



(c)



2. (a) 2π, $\frac{2\pi}{3}$, $\frac{2\pi}{5}$, ...

(b) The cosine terms; also the terms $B_n \sin nx$, n even. (In our case, $a = 2\pi$.) The function being represented has the property that $f(-x) = -f(x)$ (odd function). This property holds for $\sin nx$ but not $\cos nx$. Moreover, $f(x)$ has the property that $f(\pi - x) = f(x)$. This property does not hold for $\sin 2kx$, k integral, since $\sin 2k(\pi - x) = -\sin 2kx$. It does hold for $\sin (2k + 1)x$.

Illustrative Test Questions

1. Determine whether each of the following functions is periodic and, if so, find the fundamental period:

   (a) $y = |\cos 2x|$; (b) $y = \sin 3x \cos 3x$.

2. Given that $f\colon x \to f(x)$ is periodic with fundamental period $\frac{1}{2}$ and given that $f(\frac{1}{4}) = 2$, $f(2) = 5$, and $f(\frac{11}{8}) = 3$, find

   (a) $f(0)$; (b) $f(-\frac{5}{8})$; (c) $f(\frac{3}{4})$.

3. Sketch two complete cycles of the graph of $y = 2 \sin 3x$.

4. Change from radians to degrees:

   (a) $\frac{7\pi}{12}$; (b) $\frac{2}{15}$.

5. Change from degrees to radians:

   (a) 165°; (b) 2°.

6. What is the radius of a circle in which a sector of area 6 has a perimeter 10? (two solutions)

7. Sketch the graph of $y = \sin x - \sqrt{3} \cos x$ over a complete cycle, indicating both the fundamental period and the amplitude.

8. Express $\sin (x + 2y)$ in terms of $\sin x$, $\sin y$, $\cos x$, $\cos y$.

9. Express the following in the form $\pm\sin x$ or $\pm\cos x$:

   (a) $\sin (x + \frac{3\pi}{2})$; (c) $\sin (-3\pi - x)$;

   (b) $\cos (\frac{5\pi}{2} - x)$; (d) $\cos (x + 5\pi)$.

10. Show that

    $(\sin x + \sin 2x)(\sin x)(1 - 2 \cos x) = (\cos x + \cos 2x)(\cos 2x - \cos x)$

    holds for all real values of x.

11. Given $\sin 27^\circ = 0.4540$ and $\sin 28^\circ = 0.4695$, interpolate to find

 (a) $\sin 27.4^\circ$;

 (b) the angle between 27° and 28° whose sine is 0.4664.

12. Given the function $x \rightarrow -3 \sin (2x + \frac{\pi}{3})$, find the points on the graph with smallest positive x for which

 (a) the function has the value zero;

 (b) the function has a maximum value;

 (c) the function has a minimum value.

*13. If a, b, c are constants, find A and B such that $\sin (x + c) = A \sin (x + a) + B \sin (x + b)$ holds for all values of x. (You may assume that $\sin (a - b) \neq 0$.)

Answers to Illustrative Test Questions

1. (a) $\cos 2(x + \frac{\pi}{2}) = \cos (2x + \pi) = -\cos 2x$

 Hence, $|\cos 2(x + \frac{\pi}{2})| = |\cos 2x|$ and the period is $\frac{\pi}{2}$.

 (b) $\sin 3x \cos 3x = \frac{1}{2} \sin 6x = \frac{1}{2} \sin (6x + 2\pi) = \frac{1}{2} \sin 6(x + \frac{\pi}{3})$

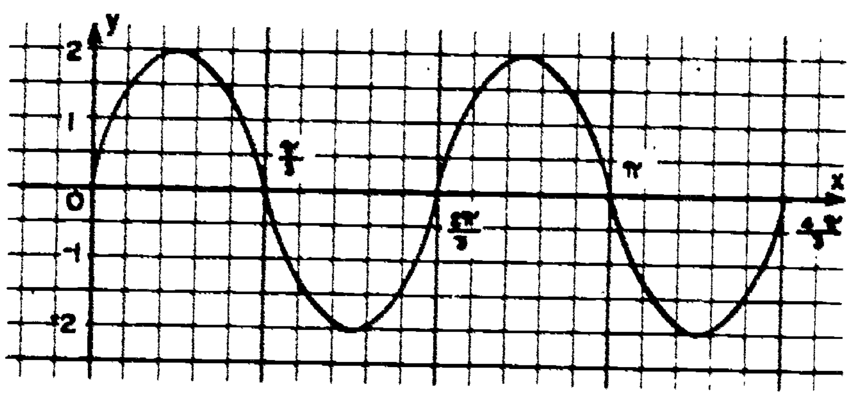
 and the period is $\frac{\pi}{3}$.

2. (a) $f(0) = f(2 - 4 \cdot \frac{1}{2}) = f(2) = 5$

 (b) $f(-\frac{5}{8}) = f(\frac{11}{8} - 4 \cdot \frac{1}{2}) = f(\frac{11}{8}) = 3$

 (c) $f(\frac{3}{4}) = f(\frac{1}{4} + \frac{1}{2}) = f(\frac{1}{4}) = 2$

3.



4. (a) $\frac{7\pi}{12} \cdot \frac{180^\circ}{\pi} = 105^\circ$ (b) $\frac{2}{15} \cdot \frac{180^\circ}{\pi} = \frac{24^\circ}{\pi}$

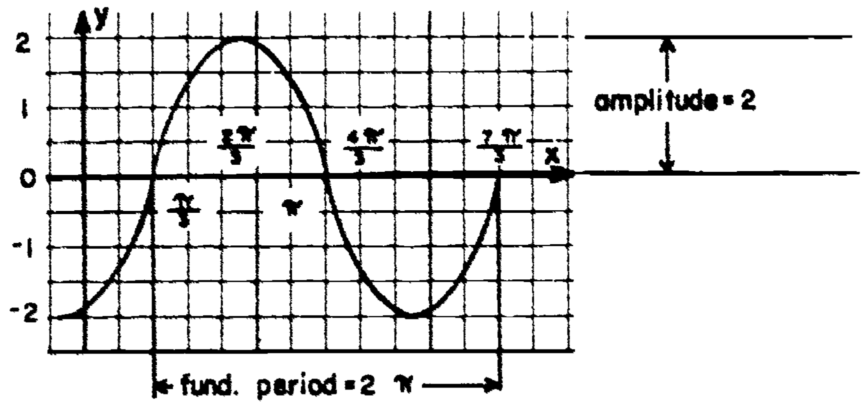
5. (a) $165^\circ \cdot \frac{\pi}{180^\circ} = \frac{11\pi}{12}$ (b) $2^\circ \cdot \frac{\pi}{180^\circ} = \frac{\pi}{90}$

6, If the radius is r and the arc s, then

$\frac{1}{2}rs = 6$; $\quad 2r + s = 10$; $\quad \frac{1}{2}r(10 - 2r) = 6$;

$5r - r^2 = 6$; $r^2 - 5r + 6 = 0$; $(r - 3)(r - 2) = 0$, $r = 3$ or 2.

7. Note that $y = \sin x - \sqrt{3}\cos x = 2\sin\left(x - \frac{\pi}{3}\right)$ for all x.



8. $\sin(x + 2y) = \sin x \cos 2y + \cos x \sin 2y$

$= \sin x (\cos^2 y - \sin^2 y) + \cos x \cdot 2 \sin y \cos y$

$= \sin x \cos^2 y - \sin x \sin^2 y + 2 \sin y \cos x \cos y$

or $\sin x - 2 \sin x \sin^2 y + 2 \sin y \cos x \cos y$

9. (a) $-\cos x$ (c) $\sin x$

(b) $\sin x$ (d) $-\cos x$

10. $(\sin x + \sin 2x)(\sin x)(1 - 2\cos x) = (\cos x + \cos 2x)(\cos 2x - \cos x)$

$= (\sin x + \sin 2x)(\sin x - 2 \sin x \cos x)$

$= (\sin x + \sin 2x)(\sin x - \sin 2x)$

$= \sin^2 x - \sin^2 2x$

$= 1 - \cos^2 x - (1 - \cos^2 2x)$

$= \cos^2 2x - \cos^2 x$

$= (\cos 2x + \cos x)(\cos 2x - \cos x)$

11. (a) $\sin 28^\circ = 0.4695$

$\sin 27^\circ = \underline{0.4540}$

0.0155

$\underline{.4}$

$.00620$

$\underline{.4540}$

$\sin 27.4^\circ = .4602$

(b) $.4664$

$\underline{.4540}$

$.0124$

$\frac{.0124}{.0155} = 0.8$

$\sin 27.8^\circ = 0.4664$

12. (a) $-3 \sin\left(2x + \frac{\pi}{3}\right) = 0$

$2x + \frac{\pi}{3} = 0, \pi, -\pi, \cdots$

$2x = -\frac{\pi}{3}, \frac{2\pi}{3}, -\frac{4\pi}{3}, \cdots$

$x = -\frac{\pi}{6}, \frac{\pi}{3}, -\frac{2\pi}{3}, \cdots$

Answer: $\frac{\pi}{3}$

(b) $-3 \sin\left(2x + \frac{\pi}{3}\right) = 3$

$\sin\left(2x + \frac{\pi}{3}\right) = -1$

$2x + \frac{\pi}{3} = \frac{3\pi}{2}, -\frac{\pi}{2}, \cdots$

$2x = \frac{7\pi}{6}, -\frac{5\pi}{6}, \cdots$

$x = \frac{7\pi}{12}, -\frac{5\pi}{12}, \cdots$

Answer: $\frac{7\pi}{12}$

(c) $-3 \sin\left(2x + \frac{\pi}{3}\right) = -3$

$\sin\left(2x + \frac{\pi}{3}\right) = 1$

$2x + \frac{\pi}{3} = \frac{\pi}{2}, -\frac{3\pi}{2}, \cdots$

$2x = \frac{\pi}{6}, -\frac{11\pi}{6}, \cdots$

$x = \frac{\pi}{12}, -\frac{11\pi}{12}, \cdots$

Answer: $\frac{\pi}{12}$

*13. If $x = -a$, we have

$\sin(c - a) = A \sin 0 + B \sin(b - a)$

$B = \frac{\sin(c - a)}{\sin(b - a)} = \frac{\sin(a - c)}{\sin(a - b)}.$

If $x = -b$, we have

$\sin(c - b) = A \sin(a - b) + B \sin 0$

$A = \frac{\sin(c - b)}{\sin(a - b)} = \frac{\sin(b - c)}{\sin(b - a)} = -\frac{\sin(b - c)}{\sin(a - b)}.$